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Free Practice

Worked Solutions

2027 Edition

Written and taught by **Leonard Shen**

USING THESE SOLUTIONS

These are the worked solutions to the **Free Practice Workbook**, in the same order with the same question numbers, with **each question printed above its solution**. **Workbook p.** beside each question is the page it sits on in the workbook, and each question shows the section of the **Notes** it practises.

M1 is a method mark and **A1** is an answer mark. **The trick**, **Watch out** and **Ways to do it** are explained in "How to use this workbook". Read them even when you got the question right: the more methods you know, the easier it is to check an answer.

Attempt the question properly before you look, and check your *method* as well as the final answer.

Calculator boxes show the TI-Nspire screen: what to type on the left, what comes back on the right. Write the equation or integral you are solving first (that is where the method mark is), and start a new problem on the calculator for every question.

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CAS commands used in these solutions

Every menu path starts from a calculator page, and any command can also be typed letter by letter. The full CAS Guide is at the back of the Notes (Topic 12).

Command	Where to find it	What we use it for
Actions and algebra		
Define or :=	menu 1 1 or ctrl + templates key	Define a function or variable to save retyping
DelVar	menu 1 3	Clear an old value from a letter (or start a new problem)
solve(eq, var)	menu 3 1	Solving; add a domain with the bar, e.g. $\|\, 0 \le x \le 2 \cdot \pi$
factor$(expr, var)$	menu 3 2	Factorising; checking proof algebra
expand$(expr, var)$	menu 3 3	Expanding; with the variable it also gives partial fractions
completeSquare$(expr, x, y)$	menu 3 5	Circles, ellipses, hyperbolas; loci in the complex plane
polyCoeffs$(poly, x)$	menu 3 8 7	Equating coefficients (e.g. a cubic with a known complex root)
tExpand, tCollect	menu 3 B	Checking trig identities (often not the form you want)
modulus $\|\square\|$	template, abs( ) or menu 2 9 5	Distance and modulus; $\|x + y \cdot i - a\|$ for loci
Calculus		
derivative $\frac{d}{d\square}\square$	menu 4 1 or shift −	Derivatives; nest it or use the second-derivative template
integral $\int_{\square}^{\square} \square\, d\square$	menu 4 3 or shift +	Definite and indefinite integrals
limit	menu 4 4	Limits and asymptotic behaviour
Σ (sum), Π (product)	menu 4 5, menu 4 6	Testing sums before an induction proof; telescoping sums
fMin, fMax	menu 4 7, menu 4 8	A check on an optimisation answer (not reliable on their own)
tangentLine, normalLine	menu 4 9, menu 4 A	Tangents and normals: tangentLine$(f(x), x, 2)$
arcLen$(expr, x, a, b)$	menu 4 B	Arc length of a curve
deSolve$(eq \text{ and } cond, x, y)$	menu 4 D, or type de then menu 3 1	Solving differential equations (use $'$ for the derivative)
impDif(eq, x, y)	menu 4 E	Implicit differentiation
euler	type it, or the catalog	Checking Euler's method
Complex numbers		
conj, real, imag, angle	menu 2 9, then 1, 2, 3 or 4	Conjugate, Re, Im, Arg; solve $\left(\text{angle}(x + y \cdot i) = \frac{\pi}{3}, y\right)$ turns a ray into a Cartesian equation
cSolve, cFactor, cZeros	menu 3 C, or type c then menu 3 1	Solving and factorising over C
Vectors		
unitV, crossP, dotP	menu 7 C, then 1, 2 or 3	Unit vectors, cross and dot products
norm	menu 7 7 1	Magnitude of a vector (not the modulus)

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Command	Where to find it	What we use it for
► Polar, ► Rect	menu 7 C 4, menu 7 C 5	Converting a vector between polar and rectangular form
Probability and statistics		
!	menu 5 1	Factorials
normCdf, binomCdf	menu 5 5 2, menu 5 5 B	Normal and binomial probabilities
invNorm	menu 5 5 3	Critical values
z Interval, z Test	menu 6 6 1, menu 6 7 1	Confidence intervals and z-tests for a mean (Stats input)

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1 Logic and Proof

Section A – Technology free

Question 1A.1* (7 marks) The language of statements

[Notes link: **1.1** The language of statements] Workbook p. 5

a. Write down the converse of "if n is a multiple of 4, then n is even", and state whether the converse is true, giving a reason. 2 marks

Converse: "if n is even, then n is a multiple of 4". It is **false**: $n = 2$ is even but not a multiple of 4. **M1 A1**

b. Write down the contrapositive of "if $x^2 > 9$, then $x > 3$ or $x < -3$". 1 mark

"If $-3 \leq x \leq 3$, then $x^2 \leq 9$." (Negate both parts and swap them. "Not ($x > 3$ or $x < -3$)" is $-3 \leq x \leq 3$.) **A1**

c. Write down the negation of "for all $n \in N$, $n^2 + n$ is even". Which of the statement and its negation is true? 2 marks

Negation: "there exists $n \in N$ such that $n^2 + n$ is odd". The original is true, since $n^2 + n = n(n+1)$ is a product of consecutive integers, one of which is even. So the negation is false. **M1 A1**

d. Show that the statement "for all $x \in R$, if $x^2 = 4$ then $x = 2$" is false. 1 mark

Counterexample: $x = -2$ gives $x^2 = 4$ but $x \neq 2$. **M1**

e. Write the statement "every real number has a real cube root" using the quantifiers $\forall$ and $\exists$. 1 mark

$\forall x \in R,\ \exists y \in R$ such that $y^3 = x$. **A1**

> **Ways to do it. 1. Converse:** swap the two halves ("if n is even then n is a multiple of 4"), then test it with a number such as $n = 2$. **2. Contrapositive:** swap *and* negate both halves; "or" becomes "and" when negated, so the new hypothesis is $-3 \leq x \leq 3$. **3. Negation:** "for all" becomes "there exists" and the claim is negated; "there exists" becomes "for all". **4. Counterexample:** one value that fails is a complete answer to "is it true?"; you do not need a general argument. **5. Check:** a statement and its contrapositive must have the same truth value, so if yours do not, one is written wrongly.

Question 1A.2** (9 marks) Mathematical induction

[Notes link: **1.6** Mathematical induction] Workbook p. 6

a. Prove by induction that $1 + 3 + 5 + \cdots + (2n-1) = n^2$ for all $n \in N$. 3 marks

Let $P(n)$: $1 + 3 + \cdots + (2n-1) = n^2$. $P(1)$: LHS $= 1 = 1^2$ = RHS. **M1**
Assume $P(k)$: $1 + 3 + \cdots + (2k-1) = k^2$. Then $1 + 3 + \cdots + (2k-1) + (2k+1) = k^2 + 2k + 1 = (k+1)^2$, which is $P(k+1)$. **M1**
Since $P(1)$ is true and $P(k) \Rightarrow P(k+1)$, $P(n)$ is true for all $n \in N$ by induction. **M1**

b. Prove by induction that $5^n - 1$ is divisible by 4 for all $n \in N$. 3 marks

$P(1)$: $5 - 1 = 4$, divisible by 4. Assume $5^k - 1 = 4m$, $m \in Z$. Then $5^{k+1} - 1 = 5 \cdot 5^k - 1 = 5(4m+1) - 1 = 20m + 4 = 4(5m+1)$, divisible by 4. True for all $n \in N$ by induction. **M1 M1 M1**

c. Prove by induction that $2^n > n$ for all $n \in N$. 3 marks

$P(1)$: $2 > 1$. Assume $2^k > k$. Then $2^{k+1} = 2 \cdot 2^k > 2k = k + k \geq k + 1$ (as $k \geq 1$). So $2^{k+1} > k + 1$, and the result holds for all $n \in N$ by induction. **M1 M1 M1**

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Ways to do it. **1. Induction:** base case, assume $P(k)$, then build $P(k+1)$: add the next term $2k+1$ to both sides in part **a.**; write $5^{k+1}-1=5(5^k-1)+4$ in part **b.**; use $2^{k+1}=2\cdot 2^k>2k\geq k+1$ in part **c.** **2. Arithmetic series:** part **a.** directly, $\frac{n}{2}(1+(2n-1))=n^2$. **3. Factorising:** $5^n-1=(5-1)(5^{n-1}+\cdots+1)$ shows the factor of 4 without induction. **4. Calculator check:** $\text{seq}(5^n-1,n,1,6)$ and Σ`(2k-1,k,1,n)` before you write the proof.

Question 1A.3*** (7 marks) Inequalities and telescoping sums

[Notes link: **1.7** Proving inequalities; **1.8** Telescoping series] Workbook p. 7

a. Prove that $a^2+b^2\geq 2ab$ for all $a,b\in R$. 1 mark

$a^2+b^2-2ab=(a-b)^2\geq 0$, so $a^2+b^2\geq 2ab$. **M1**

b. Prove that $x+\dfrac{4}{x}\geq 4$ for all $x>0$. 2 marks

For $x>0$: $x+\frac{4}{x}-4=\frac{x^2-4x+4}{x}=\frac{(x-2)^2}{x}\geq 0$, since the numerator is ≥ 0 and $x>0$. (Equality at $x=2$.) **M1 M1**

c. Express $\dfrac{1}{(2k-1)(2k+1)}$ in partial fractions. 2 marks

$\dfrac{1}{(2k-1)(2k+1)}=\dfrac{A}{2k-1}+\dfrac{B}{2k+1}\Rightarrow 1=A(2k+1)+B(2k-1)$. $k=\frac{1}{2}$: $A=\frac{1}{2}$; $k=-\frac{1}{2}$: $B=-\frac{1}{2}$. So $\dfrac{1}{2}\left(\dfrac{1}{2k-1}-\dfrac{1}{2k+1}\right)$. **M1 A1**

d. Hence show that $\displaystyle\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)}=\frac{n}{2n+1}$. 2 marks

$$\sum_{k=1}^{n}\frac{1}{2}\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right)=\frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right]$$ **M1**

$$=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)=\frac{n}{2n+1}.$$ **M1**

TI-Nspire | checks

$\text{expand}\left(\dfrac{1}{(2\cdot k-1)\cdot(2\cdot k+1)},k\right)$ $\quad\dfrac{1}{2(2k-1)}-\dfrac{1}{2(2k+1)}$

$\displaystyle\sum_{k=1}^{n}\left(\frac{1}{(2\cdot k-1)\cdot(2\cdot k+1)}\right)$ $\quad\dfrac{n}{2n+1}$

Ways to do it. **1. Perfect square:** $(a-b)^2\geq 0$ expands to the result; for part **b.** use $\left(\sqrt{x}-\frac{2}{\sqrt{x}}\right)^2\geq 0$ or multiply by $x>0$ and get $(x-2)^2\geq 0$. **2. Work backwards on scrap paper** from the result to something obviously true, then write the proof in the forward direction. **3. AM–GM:** part **b.** is part **a.** with $a=\sqrt{x}$ and $b=\frac{2}{\sqrt{x}}$. **4. Calculus:** the minimum of $x+\frac{4}{x}$ for $x>0$ is at $x=2$, value 4. **5. Telescoping:** $\frac{1}{2}\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right)$, write out the first and last terms and cancel; induction also proves the sum. **6. Calculator:** $\text{expand}(\ )$ gives the partial fractions.

Question 1A.4**** (10 marks) Induction with inequalities

[Notes link: **1.6** Mathematical induction; **1.7** Proving inequalities] Workbook p. 8

a. Let $x>-1$. Prove by induction that $(1+x)^n\geq 1+nx$ for all $n\in N$. 3 marks

$n=1$: $1+x\geq 1+x$. ✓ **M1 M1**

Assume $(1+x)^k\geq 1+kx$. Since $1+x>0$, multiplying by it keeps the inequality: $(1+x)^{k+1}\geq(1+kx)(1+x)=$

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$1+(k+1)x+kx^2 \geq 1+(k+1)x$, because $kx^2 \geq 0$. ✓ **A1**

b. Prove by induction that $2^n \geq n^2$ for all integers $n \geq 4$. 3 marks

$n=4$: $16 \geq 16$. ✓ **M1**
Assume $2^k \geq k^2$ for some $k \geq 4$. Then $2^{k+1} = 2 \times 2^k \geq 2k^2$. **M1**
Now $2k^2-(k+1)^2 = k^2-2k-1 = (k-1)^2-2 \geq 7 > 0$ for $k \geq 4$, so $2^{k+1} \geq 2k^2 > (k+1)^2$. ✓ **M1**

c. Prove by induction that $1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2} \leq 2-\frac{1}{n}$ for all $n \in N$. 4 marks

$n=1$: $1 \leq 2-1$. ✓ **M1**
Assume $1+\frac{1}{4}+\cdots+\frac{1}{k^2} \leq 2-\frac{1}{k}$. Then $1+\frac{1}{4}+\cdots+\frac{1}{k^2}+\frac{1}{(k+1)^2} \leq 2-\frac{1}{k}+\frac{1}{(k+1)^2}$. **M1**
We need this to be at most $2-\frac{1}{k+1}$, that is $\frac{1}{(k+1)^2} \leq \frac{1}{k}-\frac{1}{k+1} = \frac{1}{k(k+1)}$. **M1**
This is true because $(k+1)^2 \geq k(k+1)$. ✓
Each statement is therefore true for all the required n by induction. **M1**

The trick. With inequalities you are allowed to throw things away. After using the assumption, compare what you have with what you want by subtracting them and showing the difference is positive (part **b.**), or by working backwards to an obviously true statement (part **c.**). In part **a.** say why you may multiply by $1+x$: it is positive.

Ways to do it. **1.** Induction, then a "difference is positive" argument for the leftover inequality. **2.** Part **c.** can also be done without induction by telescoping: $\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1}-\frac{1}{k}$ for $k \geq 2$. **3.** If the base case is not $n=1$, start the induction where the statement first becomes true and say so.

Section B – Technology active

Question 1B.1*** (7 marks) Conjecture, then prove

[Notes link: **1.6** Mathematical induction; **12.6** CAS Guide: Proof] Workbook p. 9

A sequence is defined by $a_1 = 1$ and $a_{n+1} = a_n + 2n + 1$ for $n \in N$.

a. Use your calculator to find a_2, a_3, a_4, a_5. 1 mark

$a_2 = 4$, $a_3 = 9$, $a_4 = 16$, $a_5 = 25$. **A1**

b. Conjecture a formula for a_n in terms of n. 1 mark

$a_n = n^2$. **A1**

c. Prove your conjecture by mathematical induction. 3 marks

$P(1)$: $a_1 = 1 = 1^2$. Assume $a_k = k^2$. Then $a_{k+1} = a_k + 2k + 1 = k^2 + 2k + 1 = (k+1)^2$. True for all $n \in N$ by induction. **M1 M1 M1**

d. Find the smallest n such that $a_1 + a_2 + \cdots + a_n > 1000$. 2 marks

$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} > 1000$. $n = 13$ gives 819 and $n = 14$ gives 1015, so $n = 14$. **M1 A1**

TI-Nspire | Calculator

$\text{seq}(k^2, k, 1, 5)$ $\{1, 4, 9, 16, 25\}$

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$$\text{seq}\left(\sum_{k=1}^{n}\left(k^2\right), n, 12, 15\right) \qquad \{650, 819, 1015, 1240\} \quad \Rightarrow n = 14$$

Ways to do it. 1. Spot the pattern: list $1, 4, 9, 16, 25$; the differences $2n+1$ are the gaps between squares, so $a_n = n^2$. **2. Sum the differences:** $a_n = 1 + \sum_{k=1}^{n-1}(2k+1) = 1 + (n-1)(n+1) = n^2$, which is a proof without induction. **3. Induction:** $a_{k+1} = k^2 + 2k + 1 = (k+1)^2$. **4. Smallest n:** solve the inequality by hand and round up, or scroll a table, or nSolve.

Question 1B.2**** (8 marks) Factorials and induction

[Notes link: **1.6** Mathematical induction; **12.6** CAS Guide: Proof; **1.8** Telescoping series] Workbook p. 10

a. Evaluate $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4!$. 1 mark

$1 + 4 + 18 + 96 = 119$. **A1**

b. Conjecture a formula for $\sum_{k=1}^{n} k \cdot k!$. 1 mark

$1 = 2! - 1$, $5 = 3! - 1$, $23 = 4! - 1$, $119 = 5! - 1$, so $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$. **A1**

c. Prove your formula by induction. 3 marks

$P(1)$: $1 \cdot 1! = 1 = 2! - 1$. Assume $\sum_{k=1}^{m} k \cdot k! = (m+1)! - 1$. Then adding $(m+1)(m+1)!$: **M1 M1**
$(m+1)! - 1 + (m+1)(m+1)! = (m+1)!(1 + m + 1) - 1 = (m+2)! - 1$. True for all $n \in N$ by induction. **M1**

d. Prove by induction that $n! > 2^n$ for all integers $n \geq 4$. 3 marks

$P(4)$: $24 > 16$. Assume $k! > 2^k$ for some $k \geq 4$. Then $(k+1)! = (k+1)k! > (k+1)2^k > 2 \cdot 2^k = 2^{k+1}$, since $k + 1 \geq 5 > 2$. True for all $n \geq 4$ by induction. **M1 M1 M1**

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$$\sum_{k=1}^{4}(k \cdot k!) \qquad 119$$

$$\text{seq}\left(\sum_{k=1}^{n}(k \cdot k!), n, 1, 5\right) \qquad \{1, 5, 23, 119, 719\}$$

The trick. Before proving anything, notice that $k \cdot k! = (k+1)! - k!$. Once you see that, the sum telescopes and the formula is obvious. For the inequality, the induction has to start where the statement first becomes true, which is $n = 4$.

Ways to do it. 1. Small cases: $1, 5, 23, 119$ are each one less than a factorial, so conjecture $(n+1)! - 1$. **2. Induction:** $(k+1)! - 1 + (k+1)(k+1)! = (k+2)(k+1)! - 1$. **3. Telescoping:** $k \cdot k! = (k+1)! - k!$, so the sum collapses to $(n+1)! - 1!$ with no induction. **4. Induction from $n = 4$:** $(k+1)! = (k+1)k! > (k+1)2^k > 2 \cdot 2^k$ because $k + 1 > 2$; the base case must be $n = 4$, since it is false for $n = 1, 2, 3$.

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Section C – Multiple choice (technology active)

Question 1C.1** (1 mark) Setting up a contradiction

[Notes link: **1.4** Proof by contradiction] Workbook p. 11

A proof by contradiction that $\sqrt{5}$ is irrational should begin by assuming that

A. $\sqrt{5} = \dfrac{p}{q}$, where $p, q \in Z$, $q \neq 0$ and $\dfrac{p}{q}$ is in lowest terms

B. $\sqrt{5}$ is irrational

C. $\sqrt{5} = \dfrac{p}{q}$, where p and q are both odd

D. 5 is not a perfect square

Solution. Assume the opposite of what you want to prove, in a form you can work with. **A**

Question 1C.2** (1 mark) Base case

[Notes link: **1.6** Mathematical induction] Workbook p. 11

The statement $2^n > n^2$ is to be proved by induction for all integers $n \geq n_0$. The smallest valid base case n_0 is

A. 1 **B.** 3 **C.** 5 **D.** 4

Solution. $n = 4$: $16 = 16$ (not $>$). $n = 5$: $32 > 25$. ($n = 1$ works too, but it fails at $n = 2, 3, 4$, so the induction can't start there.) **C**

Question 1C.3** (1 mark) The inductive step

[Notes link: **1.6** Mathematical induction] Workbook p. 11

In proving $1 + 3 + 5 + \cdots + (2n-1) = n^2$ by induction, after assuming the result for $n = k$ you must show that

A. $1 + 3 + \cdots + (2k-1) = k^2$ **B.** $1 + 3 + \cdots + (2k+1) = k^2 + 1$

C. $1 + 3 + \cdots + (2k-1) + (2k+1) = (k+1)^2$ **D.** $2k + 1 = (k+1)^2 - k^2$ for $k = 1$

Solution. $P(k+1)$ adds the next odd number $2(k+1) - 1 = 2k + 1$, and the right side becomes $(k+1)^2$. **C**

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2 Circular Functions

Section A – Technology free

Question 2A.1* (6 marks) Exact values

[Notes link: **2.1** The unit circle; **2.5** Reciprocal circular functions; **2.4** Inverse circular functions] Workbook p. 13

Find the exact value of each of the following.

a. $\sec\left(\frac{5\pi}{6}\right)$ 1 mark

$\sec\left(\frac{5\pi}{6}\right) = \frac{1}{\cos\left(\frac{5\pi}{6}\right)} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3}$ **A1**

b. $\cot\left(-\frac{\pi}{4}\right)$ 1 mark

$\cot\left(-\frac{\pi}{4}\right) = \frac{1}{\tan\left(-\frac{\pi}{4}\right)} = -1$ **A1**

c. $\operatorname{cosec}\left(\frac{7\pi}{6}\right)$ 1 mark

$\operatorname{cosec}\left(\frac{7\pi}{6}\right) = \frac{1}{\sin\left(\frac{7\pi}{6}\right)} = \frac{1}{-\frac{1}{2}} = -2$ **A1**

d. $\sin^{-1}\left(-\frac{1}{2}\right)$ 1 mark

$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ (the range of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$) **A1**

e. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ 1 mark

$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$ (the range of $\cos^{-1}$ is $[0, \pi]$) **A1**

f. $\tan\left(\cos^{-1}\left(\frac{1}{3}\right)\right)$ 1 mark

Let $\theta = \cos^{-1}\left(\frac{1}{3}\right) \in \left(0, \frac{\pi}{2}\right)$. Draw a right triangle with adjacent 1 and hypotenuse 3, so the opposite side is $\sqrt{8} = 2\sqrt{2}$. Then $\tan(\theta) = 2\sqrt{2}$. **A1**

Ways to do it. **1. Unit circle:** find the reference angle and the quadrant, then attach the sign. **2. Reciprocal last:** for sec, cosec and cot find cos, sin or tan first, then flip it ($\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$, so $\sec = -\frac{2}{\sqrt{3}}$). **3. Inverse functions:** ask "which angle in the allowed range has this value?", $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for $\sin^{-1}$ and $[0, \pi]$ for $\cos^{-1}$. **4. Triangle:** for $\tan\left(\cos^{-1}\frac{1}{3}\right)$ draw a right-angled triangle with adjacent 1 and hypotenuse 3, so the opposite side is $2\sqrt{2}$. **5. Identity instead of a triangle:** $\tan^2\theta = \sec^2\theta - 1 = 9 - 1$. **6. Calculator** in exact mode as a check.

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Question 2A.2** (10 marks) Solving trigonometric equations

[Notes link: **2.8** Solving trigonometric equations] Workbook p. 14

Solve each equation for x over the given domain.

a. $2\sin^2(x)-\sin(x)-1=0, \quad x\in[0,2\pi]$ 3 marks

$(2\sin x+1)(\sin x-1)=0.\ \sin x=1\Rightarrow x=\frac{\pi}{2};\ \sin x=-\frac{1}{2}\Rightarrow x=\frac{7\pi}{6},\frac{11\pi}{6}.$ **M1 M1 A1**

b. $\sec^2(x)=4, \quad x\in[0,\pi]$ 2 marks

$\cos^2 x=\frac{1}{4}\Rightarrow\cos x=\pm\frac{1}{2}\Rightarrow x=\dfrac{\pi}{3},\ \dfrac{2\pi}{3}.$ **M1 A1**

c. $\sin(2x)=\cos(x), \quad x\in[0,\pi]$ 2 marks

$2\sin x\cos x-\cos x=0\Rightarrow\cos x(2\sin x-1)=0\Rightarrow x=\dfrac{\pi}{6},\ \dfrac{\pi}{2},\ \dfrac{5\pi}{6}.$ **M1 A1**

d. $2\cos^2(x)+3\sin(x)=3, \quad x\in[0,2\pi]$ 3 marks

$2(1-\sin^2 x)+3\sin x-3=0\Rightarrow 2\sin^2 x-3\sin x+1=0\Rightarrow(2\sin x-1)(\sin x-1)=0$, so $x=\dfrac{\pi}{6},\ \dfrac{\pi}{2},\ \dfrac{5\pi}{6}.$ **M1 M1 A1**

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$\text{solve}(2\cdot\sin(x)^2-\sin(x)-1=0,x)\,|\,0\le x\le 2\cdot\pi$ $\qquad x=\frac{\pi}{2}$ or $x=\frac{7\pi}{6}$ or $x=\frac{11\pi}{6}$

Ways to do it. 1. Quadratic in one function: let $a=\sin x$, factorise, and solve each factor in the domain. **2. Identity first** so only one function appears: $\cos^2 x=1-\sin^2 x$ in part **d.**, and $\sec^2 x=4\Rightarrow\cos x=\pm\frac{1}{2}$. **3. Factorise, never divide:** $\sin 2x=\cos x\Rightarrow\cos x(2\sin x-1)=0$; dividing by $\cos x$ loses $x=\frac{\pi}{2}$. **4. Complementary angles:** $\sin 2x=\sin\left(\frac{\pi}{2}-x\right)$ and use the two general cases. **5. Graph or** solve( ) with the domain as a check on the number of solutions.

Question 2A.3*** (8 marks) Identities with a trick

[Notes link: **2.7** Proving identities (show that questions)] Workbook p. 15

Prove each of the following identities.

a. $\text{cosec}(2x)-\cot(2x)=\tan(x)$ 2 marks

$\text{LHS}=\dfrac{1-\cos(2x)}{\sin(2x)}=\dfrac{2\sin^2(x)}{2\sin(x)\cos(x)}=\tan(x).$ **M1 A1**

b. $\cos^4(x)-\sin^4(x)=\cos(2x)$ 2 marks

$\text{LHS}=\left(\cos^2 x-\sin^2 x\right)\left(\cos^2 x+\sin^2 x\right)=\cos(2x)\times 1.$ **M1 A1**

c. $\sin(3x)=3\sin(x)-4\sin^3(x)$ 2 marks

$\sin(2x+x)=\sin 2x\cos x+\cos 2x\sin x=2\sin x\left(1-\sin^2 x\right)+\left(1-2\sin^2 x\right)\sin x=3\sin x-4\sin^3 x.$ **M1 A1**

d. $\sqrt{1-\sin(2a)}=\cos(a)-\sin(a),$ for $0\le a\le\dfrac{\pi}{4}$ 2 marks

$1-\sin(2a)=\cos^2 a-2\sin a\cos a+\sin^2 a=(\cos a-\sin a)^2$, so LHS $=|\cos a-\sin a|=\cos a-\sin a$, since $\cos a\ge\sin a$ on $\left[0,\frac{\pi}{4}\right]$. **M1 A1**

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The trick. **a.** With reciprocal functions, convert everything to $\sin$ and $\cos$. **b.** Look for a difference of squares. **c.** Write $3x$ as $2x+x$, and choose the form of $\cos 2x$ that matches the target ($\sin$ only). **d.** $1 \pm \sin(2a)$ is a perfect square. A square root gives a modulus, so the domain is what lets you drop it.

Ways to do it. **1. Half-angle form of the double angle formulas:** $\frac{1-\cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin x \cos x}$. **2. Difference of two squares:** $(\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)$. **3. Split the angle:** $\sin(2x+x)$, expand, then replace $\cos^2 x$ with $1-\sin^2 x$. **4. De Moivre:** the imaginary part of $(\cos x + i\sin x)^3$ gives $\sin 3x$ too. **5. Square root of a square:** $\sqrt{(\cos a - \sin a)^2} = |\cos a - \sin a|$, and the interval tells you which sign.

Question 2A.4**** (10 marks) Compositions of inverse functions

[Notes link: **2.4** Inverse circular functions; **2.6** Trigonometric identities] Workbook p. 16

a. Show that $\sin\left(2\cos^{-1}(x)\right) = 2x\sqrt{1-x^2}$ for $x \in [-1,1]$. 2 marks

Let $\theta = \cos^{-1}(x)$, so $\cos\theta = x$ and $\theta \in [0,\pi]$, which makes $\sin\theta = \sqrt{1-x^2} \geq 0$. **M1**

$\sin(2\theta) = 2\sin\theta\cos\theta = 2x\sqrt{1-x^2}$. **M1**

b. Express $\cos\left(\tan^{-1}(x)\right)$ and $\sin\left(2\tan^{-1}(x)\right)$ in terms of x, without any circular functions. 3 marks

Let $\theta = \tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so $\cos\theta > 0$ and $\sec^2\theta = 1+x^2$. **M1**

$\cos\left(\tan^{-1}(x)\right) = \dfrac{1}{\sqrt{1+x^2}}, \quad \sin\theta = \tan\theta\cos\theta = \dfrac{x}{\sqrt{1+x^2}},$ **M1**

$\sin\left(2\tan^{-1}(x)\right) = 2\sin\theta\cos\theta = \dfrac{2x}{1+x^2}.$ **A1**

c. Find the maximal domain and the range of $f(x) = \sin^{-1}\left(x^2-1\right)$. 2 marks

Need $-1 \leq x^2-1 \leq 1 \iff 0 \leq x^2 \leq 2$: domain $\left[-\sqrt{2}, \sqrt{2}\right]$. **M1**

On this domain x^2-1 takes every value in $[-1,1]$, so the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. **A1**

d. Solve $2\sin^{-1}(x) = \cos^{-1}(x)$. 3 marks

Let $\theta = \sin^{-1}(x)$. Taking the cosine of both sides of $2\theta = \cos^{-1}(x)$: $\cos(2\theta) = x$, **M1**

$1-2x^2 = x \Rightarrow 2x^2+x-1 = 0 \Rightarrow (2x-1)(x+1) = 0 \Rightarrow x = \frac{1}{2}$ or $x = -1$. **M1**

Check: $x = \frac{1}{2}$ gives $2 \times \frac{\pi}{6} = \frac{\pi}{3}$. ✓ $x = -1$ gives $-\pi$ on the left and π on the right. ×

So $x = \dfrac{1}{2}$. **A1**

The trick. Taking sin, cos or tan of both sides can create answers that do not work, in the same way squaring does. Always substitute back into the original equation. Here $x = -1$ survives the algebra but fails the check.

Ways to do it. **1.** Name the inverse function θ and state its range, so you know the signs. **2.** Draw a triangle with sides x, 1 and $\sqrt{1 \pm x^2}$. **3.** For equations, apply a circular function to both sides and then check. **4.** CAS: $\text{solve}(2 \cdot \sin^{-1}(x) = \cos^{-1}(x), x)$ gives $x = \frac{1}{2}$ directly, and tExpand( ) confirms the identities.

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Section B – Technology active

Question 2B.1*** (8 marks) Inverse tangent and its inverse

[Notes link: **2.4** Inverse circular functions; **12.7** CAS Guide: Circular functions] Workbook p. 17

Let $f\colon R \to R$, $f(x) = 2\tan^{-1}(x-1)+\pi$.

a. State the range of f and the equations of its asymptotes. 2 marks

$\tan^{-1}(x-1) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so $f(x) \in (0, 2\pi)$. Range $(0, 2\pi)$; asymptotes $y = 0$ and $y = 2\pi$. **M1 A1**

b. Find the rule and domain of f^{-1}. 3 marks

Swap: $x = 2\tan^{-1}(y-1)+\pi \Rightarrow \tan^{-1}(y-1) = \frac{x-\pi}{2} \Rightarrow y = 1+\tan\left(\dfrac{x-\pi}{2}\right)$. **M1 M1**

$f^{-1}\colon (0, 2\pi) \to R,\ f^{-1}(x) = 1+\tan\left(\dfrac{x-\pi}{2}\right)$. **A1**

c. Explain why every point of intersection of the graphs of f and f^{-1} lies on the line $y = x$. 1 mark

Suppose (a, b) is on both graphs, so $f(a) = b$ and $f(b) = a$. If $a < b$, then f strictly increasing gives $f(a) < f(b)$, i.e. $b < a$: a contradiction. Similarly $a > b$ is impossible. So $a = b$, and the point is on $y = x$. **A1**

d. Find the coordinates of the point of intersection of the graphs of f and f^{-1}, correct to three decimal places. 2 marks

Solve $f(x) = x$: $x \approx 5.879$. The point is $(5.879,\ 5.879)$. **M1 A1**

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Define $f(x) = 2\cdot\tan^{-1}(x-1)+\pi$	
Define $g(x) = 1+\tan\left(\dfrac{x-\pi}{2}\right)$	
$f(g(x))\,\vert\, 0 < x < 2\cdot\pi$	x (confirms $g = f^{-1}$)
$\text{nSolve}(f(x) = x, x)$	$5.87885\ldots$

Ways to do it. 1. Transform the range and asymptotes of $\tan^{-1}$: $(0, 2\pi)$, $y = 0$ and $y = 2\pi$. **2. Swap and solve:** $x = 2\tan^{-1}(y-1)+\pi \Rightarrow y = 1+\tan\left(\frac{x-\pi}{2}\right)$, with the domain equal to the range of f. **3. Increasing function:** if f is increasing, $f(a) = b$ and $f(b) = a$ with $a < b$ is impossible, so intersections with f^{-1} are on $y = x$. **4. Calculator:** $\text{solve}(f(x) = x, x)$ or nSolve with a starting value.

Question 2B.2***** (11 marks) A function in disguise

[Notes link: **2.4** Inverse circular functions; **5.1** Differentiation formulae; **12.7** CAS Guide: Circular functions] Workbook p. 18

Let $f\colon [-1, 1] \to R$, $f(x) = \sin^{-1}(x)+\sin^{-1}\left(\sqrt{1-x^2}\right)$.

a. Find $f(0)$, $f\left(\dfrac{1}{2}\right)$ and $f(1)$. 2 marks

$f(0) = 0+\sin^{-1}(1) = \dfrac{\pi}{2},\ f\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}+\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{6}+\dfrac{\pi}{3} = \dfrac{\pi}{2},\ f(1) = \dfrac{\pi}{2}+0 = \dfrac{\pi}{2}$. **M1 A1**

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b. Show that $f'(x) = 0$ for $0 < x < 1$, and hence write down the rule of f for $0 \le x \le 1$. 3 marks

$\frac{d}{dx}\sin^{-1}\left(\sqrt{1-x^2}\right) = \frac{1}{\sqrt{1-(1-x^2)}} \times \frac{-x}{\sqrt{1-x^2}} = \frac{-x}{\sqrt{x^2}\sqrt{1-x^2}} = \frac{-x}{|x|\sqrt{1-x^2}}$. **M1**

For $0 < x < 1$, $|x| = x$, so this is $-\frac{1}{\sqrt{1-x^2}}$ and $f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$. **M1**

So f is constant on $[0,1]$: $f(x) = \frac{\pi}{2}$. **M1**

c. Find $f\left(-\frac{1}{2}\right)$ and $f(-1)$. What does this tell you about your answer to part **b.**? 2 marks

$f\left(-\frac{1}{2}\right) = -\frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{6}$ and $f(-1) = -\frac{\pi}{2} + 0 = -\frac{\pi}{2}$. So f is *not* $\frac{\pi}{2}$ for negative x: the result in part **b.** only holds where $|x| = x$. **M1 A1**

d. Find $f'(x)$ for $-1 < x < 0$, and hence write f as a hybrid function. 3 marks

For $-1 < x < 0$, $|x| = -x$, so the second derivative term is $+\frac{1}{\sqrt{1-x^2}}$ and $f'(x) = \frac{2}{\sqrt{1-x^2}}$. **M1 M1**

Then $f(x) = 2\sin^{-1}(x) + c$, and $f(-1) = -\frac{\pi}{2}$ gives $c = \frac{\pi}{2}$.

$$f(x) = \begin{cases} 2\sin^{-1}(x) + \frac{\pi}{2}, & -1 \le x < 0 \\ \frac{\pi}{2}, & 0 \le x \le 1 \end{cases}$$

A1

e. State the range of f. 1 mark

$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **A1**

Watch out. $\sqrt{x^2} = |x|$, not x. If you cancel it as x you "prove" that f is constant on the whole domain, and part **c.** shows that is false.

The trick. Evaluate before you differentiate: three equal values suggest a constant, and one value on the other side of zero tells you the pattern breaks there. A zero derivative gives a constant only on an interval where the working is valid.

Ways to do it. **1. Differentiate** and track the modulus. **2. Triangle:** for $x \ge 0$, $\sin^{-1}(x)$ and $\sin^{-1}\left(\sqrt{1-x^2}\right)$ are the two acute angles of a right-angled triangle with sides x, $\sqrt{1-x^2}$, 1. **3. Substitute** $x = \sin\theta$**:** $\sqrt{1-x^2} = \cos\theta$ and $\sin^{-1}(\cos\theta) = \frac{\pi}{2} - |\theta|$. **4. Graph it** on the calculator to see the hybrid shape straight away.

Section C – Multiple choice (technology active)

Question 2C.1** (1 mark) Maximal domain

[Notes link: **2.4** Inverse circular functions] Workbook p. 20

The maximal domain of $y = \cos^{-1}\left(\frac{1}{x}\right)$ is

A. $[-1,0) \cup (0,1]$ **B.** $R \setminus \{0\}$ **C.** $(-\infty,-1] \cup [1,\infty)$ **D.** $[1,\infty)$

Solution. Need $-1 \le \frac{1}{x} \le 1$, i.e. $|x| \ge 1$. **C**

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Question 2C.2*** (1 mark) Sum of inverse tangents

[Notes link: **2.4** Inverse circular functions; **2.6** Trigonometric identities] Workbook p. 20

$\tan^{-1}(2)+\tan^{-1}(3)$ is equal to

A. $\frac{\pi}{2}$ **B.** $-\frac{\pi}{4}$ **C.** $\frac{3\pi}{4}$ **D.** $\tan^{-1}(5)$

Solution. $\tan(\text{sum}) = \frac{5}{1-6} = -1$, and both angles are in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, so the sum is in $\left(\frac{\pi}{2}, \pi\right)$: $\frac{3\pi}{4}$. **C**

Question 2C.3*** (1 mark) Transformed secant

[Notes link: **2.5** Reciprocal circular functions; **2.3** Sketching transformed graphs] Workbook p. 20

The graph of $y = a\sec(x)+b$ has turning points at $(0,3)$ and $(\pi,-1)$. The values of a and b are

A. $a=3, b=0$ **B.** $a=2, b=1$ **C.** $a=1, b=2$ **D.** $a=-2, b=1$

Solution. $\sec 0 = 1$ and $\sec\pi = -1$: $a+b=3$ and $-a+b=-1$, so $a=2$, $b=1$. **B**

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3 Vectors, Lines and Planes

Section A – Technology free

Question 3A.1* (7 marks) Vector basics

[Notes link: **3.1** What is a vector?; **3.2** Operations on vectors; **3.4** The dot (scalar) product; **3.16** Where the dot product comes from ◇; **3.5** The angle between two vectors; **3.6** Vector resolutes (projections)] Workbook p. 23

Let $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$.

a. Find $|\underset{\sim}{a}|$. 1 mark

$|\underset{\sim}{a}| = \sqrt{4+1+4} = 3$ **A1**

b. Find the unit vector in the direction of $\underset{\sim}{b}$. 1 mark

$|\underset{\sim}{b}| = 3$, so $\hat{\underset{\sim}{b}} = \frac{1}{3}(\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k})$. **A1**

c. Find $\underset{\sim}{a} \cdot \underset{\sim}{b}$. 1 mark

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 2 - 2 + 4 = 4$ **A1**

d. Find the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$, in exact form. 1 mark

$\cos\theta = \frac{4}{3\times 3} \Rightarrow \theta = \cos^{-1}\left(\frac{4}{9}\right)$ **A1**

e. Find the vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$. 2 marks

$(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|^2}\underset{\sim}{b} = \frac{4}{9}(\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k})$ **M1 A1**

f. Hence find the vector resolute of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$. 1 mark

$\underset{\sim}{a} - \frac{4}{9}\underset{\sim}{b} = \frac{1}{9}(14\underset{\sim}{i} - 17\underset{\sim}{j} + 10\underset{\sim}{k})$ **A1**

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$a := \begin{bmatrix} 2 & -1 & 2 \end{bmatrix} : b := \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$

$\frac{\text{dotP}(a,b)}{\text{norm}(b)^2} \cdot b$ $\quad \left[\frac{4}{9},\ \frac{8}{9},\ \frac{8}{9}\right]$

$a - \frac{\text{dotP}(a,b)}{\text{norm}(b)^2} \cdot b$ $\quad \left[\frac{14}{9},\ -\frac{17}{9},\ \frac{10}{9}\right]$

Ways to do it. 1. Component formulas: $|\underset{\sim}{a}| = \sqrt{4+1+4} = 3$, unit vector $= \underset{\sim}{b} \div |\underset{\sim}{b}|$, dot product by matching components. **2. Angle:** $\cos\theta = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$, and recognise the exact value. **3. Vector resolute:** $\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}\cdot\underset{\sim}{b}}\underset{\sim}{b}$, or (scalar resolute) $\times$ (unit vector); the perpendicular part is $\underset{\sim}{a}$ minus this, and a dot product of zero with $\underset{\sim}{b}$ checks it. **4. Calculator:** norm, unitV, dotP (menu 7 C).

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Question 3A.2** (9 marks) Lines in three dimensions

[Notes link: **3.9** Equations of lines] Workbook p. 24

The line ℓ passes through $A(1,-2,3)$ and $B(3,0,2)$.

a. Find a vector equation of ℓ. 2 marks

$\overrightarrow{AB} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, so $\underset{\sim}{r}(t) = (\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}) + t(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}), t \in R$. **M1 A1**

b. Write ℓ in Cartesian form. 1 mark

$\dfrac{x-1}{2} = \dfrac{y+2}{2} = \dfrac{z-3}{-1}$ **A1**

c. Determine whether the point $(5,2,2)$ lies on ℓ. 1 mark

$x = 5 \Rightarrow t = 2 \Rightarrow$ the point is $(5,2,1) \neq (5,2,2)$. Not on ℓ. **A1**

d. Find the coordinates of the point where ℓ crosses the xy-plane. 2 marks

$z = 0 \Rightarrow 3 - t = 0 \Rightarrow t = 3$: $(7,4,0)$. **M1 A1**

e. Find the coordinates of the point on ℓ closest to the origin. 3 marks

The closest point has $\underset{\sim}{r}(t) \cdot \underset{\sim}{d} = 0$ with $\underset{\sim}{d} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$: $2(1+2t) + 2(-2+2t) - (3-t) = 9t - 5 = 0 \Rightarrow t = \frac{5}{9}$.
Point $\left(\frac{19}{9}, -\frac{8}{9}, \frac{22}{9}\right)$. **M1 M1 A1**

Ways to do it. 1. Vector equation: $\underset{\sim}{r} = \underset{\sim}{a} + t(\underset{\sim}{b} - \underset{\sim}{a})$. **2. Cartesian form:** solve each coordinate for t and set them equal. **3. Point on a line:** find t from one coordinate and test the other two. **4. Crossing a coordinate plane:** set that coordinate to zero and solve for t. **5. Closest point:** $(\underset{\sim}{r}(t) - \underset{\sim}{p}) \cdot \underset{\sim}{d} = 0$, or minimise the distance squared with calculus, or use a resolute: $t = \frac{\overrightarrow{AP} \cdot \underset{\sim}{d}}{|\underset{\sim}{d}|^2}$.

Question 3A.3*** (8 marks) Intersections and parallel lines

[Notes link: **3.10** Equations of planes] Workbook p. 25

Consider the planes $\Pi_1 : x + y - z = 1$ and $\Pi_2 : 2x - y + z = 5$.

a. Show that Π_1 and Π_2 are perpendicular. 1 mark

$\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2 = (1)(2) + (1)(-1) + (-1)(1) = 0$, so the normals (and the planes) are perpendicular. **M1**

b. Find a vector equation of the line of intersection of Π_1 and Π_2. 3 marks

Direction $\underset{\sim}{n}_1 \times \underset{\sim}{n}_2 = 0\underset{\sim}{i} - 3\underset{\sim}{j} - 3\underset{\sim}{k}$, i.e. $\underset{\sim}{j} + \underset{\sim}{k}$. A common point: set $z = 0$: $x + y = 1, 2x - y = 5 \Rightarrow x = 2,\ y = -1$. **M1 M1**

$\underset{\sim}{r}(t) = (2\underset{\sim}{i} - \underset{\sim}{j}) + t(\underset{\sim}{j} + \underset{\sim}{k}), t \in R$. (Any point on the line is fine, e.g. $(2,0,1)$.) **A1**

c. Show that the line $\underset{\sim}{r}(s) = (\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}) + s(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k})$ lies in Π_1. 2 marks

Direction: $(1)(1) + (1)(2) + (-1)(3) = 0$, so the line is parallel to Π_1. Point $(1,1,1)$: $1 + 1 - 1 = 1$, so it is in Π_1. Hence the whole line lies in Π_1. **M1 M1**

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d. The line $\underset{\sim}{r}(s) = (\underset{\sim}{j} + 2\underset{\sim}{k}) + s(\underset{\sim}{i} + k\underset{\sim}{j} + 2\underset{\sim}{k})$ is parallel to Π_2. Find k, and show that the line does not lie in Π_2. 2 marks

Parallel: $\underset{\sim}{n_2} \cdot \underset{\sim}{d} = 2 - k + 2 = 0 \Rightarrow k = 4$. The point $(0,1,2)$ gives $0 - 1 + 2 = 1 \neq 5$, so the line is not in Π_2. **M1 A1**

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$\text{crossP}\left(\begin{bmatrix}1 & 1 & -1\end{bmatrix}, \begin{bmatrix}2 & -1 & 1\end{bmatrix}\right)$	$[0,\ -3,\ -3]$
$\text{solve}(x+y-z=1 \text{ and } 2\cdot x-y+z=5, \{x,y\})$	$x=2 \text{ and } y=z-1$

Ways to do it. 1. Perpendicular planes: $\underset{\sim}{n_1} \cdot \underset{\sim}{n_2} = 0$. **2. Line of intersection by cross product:** direction $\underset{\sim}{n_1} \times \underset{\sim}{n_2}$ plus one common point (set one variable to 0). **3. By elimination:** add the equations to get x, then let $z = t$. **4. A line in a plane:** its direction is perpendicular to the normal *and* one of its points satisfies the plane; or substitute the whole line and get an identity in the parameter. **5. Calculator:** $\text{solve}(\ )$ for the system with a parameter.

Question 3A.4**** (10 marks) Where two lines in a triangle meet

[Notes link: **3.7** Collinearity; **3.3** Linear dependence and independence; **3.14** Vector proofs] Workbook p. 26

In triangle OAB, $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$. M is the midpoint of OA, and N is the point on AB such that $AN : NB = 1 : 2$. The lines ON and BM meet at P.

a. Express $\overrightarrow{ON}$ and $\overrightarrow{BM}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. 2 marks

$\overrightarrow{ON} = \underset{\sim}{a} + \frac{1}{3}(\underset{\sim}{b} - \underset{\sim}{a}) = \frac{2}{3}\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{b}$, $\quad \overrightarrow{BM} = \frac{1}{2}\underset{\sim}{a} - \underset{\sim}{b}$ **M1 A1**

b. By writing $\overrightarrow{OP}$ in two different ways, find $\overrightarrow{OP}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. 4 marks

P is on ON: $\overrightarrow{OP} = \lambda\left(\frac{2}{3}\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{b}\right)$. P is on BM: $\overrightarrow{OP} = \underset{\sim}{b} + \mu\left(\frac{1}{2}\underset{\sim}{a} - \underset{\sim}{b}\right)$. **M1 M1**

$\underset{\sim}{a}$ and $\underset{\sim}{b}$ are not parallel, so the coefficients must match: $\frac{2}{3}\lambda = \frac{1}{2}\mu$ and $\frac{1}{3}\lambda = 1 - \mu$. **M1**

$\mu = \frac{4}{3}\lambda \Rightarrow \frac{1}{3}\lambda = 1 - \frac{4}{3}\lambda \Rightarrow \lambda = \frac{3}{5}, \mu = \frac{4}{5}$. $\quad \overrightarrow{OP} = \frac{2}{5}\underset{\sim}{a} + \frac{1}{5}\underset{\sim}{b}$ **A1**

c. Find the ratios $OP : PN$ and $BP : PM$. 2 marks

$\lambda = \frac{3}{5}$: $OP : PN = 3 : 2$. $\quad \mu = \frac{4}{5}$: $BP : PM = 4 : 1$. **M1 A1**

d. Find the ratio of the area of triangle OPB to the area of triangle OAB. 2 marks

Area $OPB = \frac{1}{2}\left|\overrightarrow{OP} \times \underset{\sim}{b}\right| = \frac{1}{2}\left|\frac{2}{5}\underset{\sim}{a} \times \underset{\sim}{b}\right| = \frac{2}{5} \times \frac{1}{2}|\underset{\sim}{a} \times \underset{\sim}{b}|$, so the ratio is $2 : 5$. **M1 A1**

The trick. Write the position vector of the meeting point twice, once along each line, with two different parameters. Because $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are linearly independent you may equate the coefficients of $\underset{\sim}{a}$ and of $\underset{\sim}{b}$. In part **d.** remember $\underset{\sim}{b} \times \underset{\sim}{b} = \underset{\sim}{0}$, which kills half the working.

Ways to do it. 1. Two parameters and equating coefficients (the standard method). **2.** Put the triangle in coordinates, for example $O(0,0)$, $A(1,0)$, $B(0,1)$, and intersect the two lines: ratios are not changed by this. **3.** For area ratios use the cross product, or compare bases and heights of triangles sharing a vertex.

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Section B – Technology active

Question 3B.1*** (10 marks) A rectangle in space

[Notes link: **3.4** The dot (scalar) product; **3.10** Equations of planes; **3.13** Areas and volumes; **12.8** CAS Guide: Vectors, lines and planes] Workbook p. 27

Consider the points $A(2,1,0)$, $B(4,3,1)$ and $C(c,0,3)$, where $c \in R$.

a. Find c such that $\angle ABC = 90°$. 2 marks

$\overrightarrow{BA} = -2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$, $\overrightarrow{BC} = (c-4)\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$. $\overrightarrow{BA} \cdot \overrightarrow{BC} = -2(c-4) + 6 - 2 = 0 \Rightarrow c = 6$. **M1 A1**

b. Find the area of triangle ABC. 2 marks

$|\overrightarrow{BA}| = 3$, $|\overrightarrow{BC}| = \sqrt{4+9+4} = \sqrt{17}$, area $= \frac{1}{2} \times 3 \times \sqrt{17} = \dfrac{3\sqrt{17}}{2} \approx 6.18$. **M1 A1**

c. Find the coordinates of D such that $ABCD$ is a rectangle. 2 marks

$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{BC} = (2,1,0) + (2,-3,2)$, so $D(4,-2,2)$. **M1 A1**

d. Find the Cartesian equation of the plane containing $ABCD$. 2 marks

$\underset{\sim}{n} = \overrightarrow{BA} \times \overrightarrow{BC} = -7\underset{\sim}{i} + 2\underset{\sim}{j} + 10\underset{\sim}{k}$. Through A: $-7x + 2y + 10z = -12$, i.e. $7x - 2y - 10z = 12$. **M1 A1**

e. Find the acute angle between this plane and the xy-plane, correct to the nearest tenth of a degree. 2 marks

The angle between the planes is the angle between the normals $\underset{\sim}{n}$ and $\underset{\sim}{k}$: $\cos\theta = \dfrac{10}{\sqrt{153}} \Rightarrow \theta \approx 36.1°$. **M1 A1**

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$a := \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}$: $b := \begin{bmatrix} 4 & 3 & 1 \end{bmatrix}$

$\text{solve}\left(\text{dotP}\left(a-b, \begin{bmatrix} c & 0 & 3 \end{bmatrix} - b\right) = 0, c\right)$ $\quad c = 6$

$n := \text{crossP}\left(a-b, \begin{bmatrix} 6 & 0 & 3 \end{bmatrix} - b\right)$ $\quad [-7,\ 2,\ 10]$

$\text{dotP}\left(n, \begin{bmatrix} x & y & z \end{bmatrix}\right) = \text{dotP}(n,a)$ $\quad -7x + 2y + 10z = -12$

$\cos^{-1}\left(\dfrac{10}{\text{norm}(n)}\right)$

in degree mode: $36.055\ldots°$

Ways to do it. 1. Right angle: $\overrightarrow{BA} \cdot \overrightarrow{BC} = 0$ and solve for c. **2. Area:** $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{BC}|$, or half base times height because the angle at B is $90°$. **3. Fourth vertex:** $D = A + \overrightarrow{BC}$ (check with the midpoints of the diagonals). **4. Plane:** normal $\overrightarrow{AB} \times \overrightarrow{BC}$. **5. Angle between planes:** the acute angle between the normals. **6. Calculator:** dotP, crossP, norm.

Question 3B.2***** (10 marks) The best plane through a line

[Notes link: **3.10** Equations of planes; **3.11** The angle between a line and a plane; **3.6** Vector resolutes (projections)] Workbook p. 28

The line ℓ has equation $\underset{\sim}{r}(t) = (\underset{\sim}{i} + 2\underset{\sim}{k}) + t(2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k})$, and the plane Π_0 has equation $x + 2y + 2z = 0$. Let $\underset{\sim}{d} = 2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{m} = \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$. The aim is to find the plane that contains ℓ and makes the smallest possible acute angle with Π_0.

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a. A plane Π with normal $\underset{\sim}{n}$ contains ℓ. Explain why $\underset{\sim}{n} \cdot \underset{\sim}{d} = 0$. 1 mark

$\underset{\sim}{d}$ is the direction of a line lying in Π, and the normal is perpendicular to every direction in the plane. **A1**

b. Find the acute angle between ℓ and Π_0, in degrees correct to two decimal places. 2 marks

$$\sin\phi = \frac{|\underset{\sim}{d} \cdot \underset{\sim}{m}|}{|\underset{\sim}{d}||\underset{\sim}{m}|} = \frac{|2-2+4|}{3\times 3} = \frac{4}{9} \Rightarrow \phi = 26.39^\circ$$ **M1 A1**

c. Let $\underset{\sim}{n} = \underset{\sim}{m} - \lambda \underset{\sim}{d}$. Find the value of λ for which $\underset{\sim}{n}$ is perpendicular to $\underset{\sim}{d}$, and hence show that the plane with this normal that contains ℓ is $x + 22y + 10z = 21$. 3 marks

$(\underset{\sim}{m} - \lambda \underset{\sim}{d}) \cdot \underset{\sim}{d} = 0 \Rightarrow 4 - 9\lambda = 0 \Rightarrow \lambda = \frac{4}{9}$. **M1 M1**

$\underset{\sim}{n} = \underset{\sim}{m} - \frac{4}{9}\underset{\sim}{d} = \frac{1}{9}(\underset{\sim}{i} + 22\underset{\sim}{j} + 10\underset{\sim}{k})$, so use $\underset{\sim}{i} + 22\underset{\sim}{j} + 10\underset{\sim}{k}$. Through $(1,0,2)$: $x + 22y + 10z = 1 + 0 + 20 = 21$. **A1**

d. Find the acute angle between this plane and Π_0, in degrees correct to two decimal places. 2 marks

$\cos\theta = \frac{|1+44+20|}{\sqrt{585}\times 3} = \frac{65}{3\sqrt{585}} \Rightarrow \theta = 26.39^\circ$, the same as part **b.** **M1 A1**

e. Explain why no plane that contains ℓ can make a smaller angle with Π_0 than your answer to part **b.**, and hence why the plane in part **c.** is the one required. 2 marks

The angle between two planes is the *largest* angle that any line in one of them makes with the other (it is reached by lines perpendicular to the line where the planes meet). ℓ lies in Π and makes 26.39° with Π_0, so the angle between Π and Π_0 is at least 26.39° for every plane Π containing ℓ. The plane in part **c.** reaches exactly this value, so it gives the minimum. **M1 A1**

The trick. The normal you want is $\underset{\sim}{m}$ with its component along $\underset{\sim}{d}$ removed: the part of $\underset{\sim}{m}$ perpendicular to the line. That is a vector resolute. Among all normals perpendicular to $\underset{\sim}{d}$, it is the one closest in direction to $\underset{\sim}{m}$, which is the same as making the angle between the planes as small as possible.

Ways to do it. 1. Remove the resolute: $\underset{\sim}{n} = \underset{\sim}{m} - \frac{\underset{\sim}{m}\cdot\underset{\sim}{d}}{|\underset{\sim}{d}|^2}\underset{\sim}{d}$. **2.** Equivalent cross products: $\underset{\sim}{n} = \underset{\sim}{d} \times (\underset{\sim}{m} \times \underset{\sim}{d})$ gives the same direction. **3.** Calculus: write a general normal perpendicular to $\underset{\sim}{d}$ with one parameter, express $\cos\theta$ and maximise it with fMax. **4.** Check with the fact that the minimum angle equals the angle between the line and Π_0.

Section C – Multiple choice (technology active)

Question 3C.1** (1 mark) Parallel planes

[Notes link: **3.10** Equations of planes] Workbook p. 30

The planes $x + 2y - z = 3$ and $2x + 4y + kz = 1$ are parallel when k equals

A. -2 **B.** 2 **C.** $-\frac{1}{2}$ **D.** 1

Solution. The normals must be parallel: $(2,4,k) = 2(1,2,-1)$. **A**

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Question 3C.2*** (1 mark) Resolutes in both directions

[Notes link: **3.6** Vector resolutes (projections); **3.4** The dot (scalar) product] Workbook p. 30

Let $\underset{\sim}{s} = 3\,\underset{\sim}{i} - 4\,\underset{\sim}{k}$ and let $\underset{\sim}{r}$ be a vector with $|\underset{\sim}{r}| = 25$. The vector resolute of $\underset{\sim}{r}$ in the direction of $\underset{\sim}{s}$ is $9\,\underset{\sim}{i} - 12\,\underset{\sim}{k}$. The scalar resolute of $\underset{\sim}{s}$ in the direction of $\underset{\sim}{r}$ is

A. 3 **B.** 5 **C.** 15 **D.** $\dfrac{3}{5}$

Solution. The vector resolute is $\dfrac{\underset{\sim}{r}\cdot\underset{\sim}{s}}{|\underset{\sim}{s}|^2}\,\underset{\sim}{s} = 3\,\underset{\sim}{s}$, and $|\underset{\sim}{s}|^2 = 25$, so $\underset{\sim}{r}\cdot\underset{\sim}{s} = 75$. The scalar resolute of $\underset{\sim}{s}$ in the direction of $\underset{\sim}{r}$ is $\dfrac{\underset{\sim}{r}\cdot\underset{\sim}{s}}{|\underset{\sim}{r}|} = \dfrac{75}{25} = 3$. (15 is the scalar resolute the other way round, 5 is $|\underset{\sim}{s}|$ and $\frac{3}{5}$ is the cosine of the angle between them.) The dot product is the link between the two resolutes: find it from one and use it in the other. **A**

Question 3C.3*** (1 mark) Changing the constant

[Notes link: **3.10** Equations of planes; **3.12** Shortest distances] Workbook p. 30

The plane $2x + y - z = 5$ is moved, without being tilted, until its equation is $2x + y - z = -1$. The plane has moved

A. $\sqrt{6}$ units in the direction of $2\,\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ **B.** $\sqrt{6}$ units in the direction of $-2\,\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$

C. 6 units in the direction of $2\,\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ **D.** 6 units in the direction of $-2\,\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$

Solution. The constant has gone down by 6. Decreasing the constant moves a plane *against* its normal $2\,\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$, that is, in the direction of $-2\,\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$. The distance is $\dfrac{|\Delta k|}{|\underset{\sim}{n}|} = \dfrac{6}{\sqrt{6}} = \sqrt{6}$, not 6: the change in the constant is only the distance when the normal is a unit vector. **B**

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4 Complex Numbers

Section A – Technology free

Question 4A.1* (7 marks) Polar form

[Notes link: **4.3** Polar form] Workbook p. 33

a. Express $-1+\sqrt{3}i$ in polar form, using the principal argument. 2 marks

$r=\sqrt{1+3}=2$. The point is in quadrant 2 with reference angle $\frac{\pi}{3}$, so $\theta=\frac{2\pi}{3}$: $2\text{cis}\left(\frac{2\pi}{3}\right)$. **M1 A1**

b. Express $4\text{cis}\left(-\frac{3\pi}{4}\right)$ in Cartesian form. 1 mark

$4\left(\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right)=-2\sqrt{2}-2\sqrt{2}i$ **A1**

c. Let $z=2\text{cis}\left(\frac{\pi}{3}\right)$ and $w=\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$. Find zw and $\frac{z}{w}$ in polar form. 2 marks

$zw=2\sqrt{2}\text{cis}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=2\sqrt{2}\text{cis}\left(\frac{\pi}{12}\right)$; $\frac{z}{w}=\frac{2}{\sqrt{2}}\text{cis}\left(\frac{\pi}{3}+\frac{\pi}{4}\right)=\sqrt{2}\text{cis}\left(\frac{7\pi}{12}\right)$. **M1 A1**

d. Find $(1+i)^8$. 2 marks

$1+i=\sqrt{2}\text{cis}\,\frac{\pi}{4}$, so $(1+i)^8=(\sqrt{2})^8\text{cis}(2\pi)=16$. **M1 A1**

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$(-1+\sqrt{3}\cdot i)\blacktriangleright\text{Polar}$	$2e^{\frac{2\pi}{3}i}$
$(1+i)^8$	16

Ways to do it. **1. Mini sketch:** modulus by Pythagoras, argument from the reference angle and the quadrant. **2. Unit circle:** divide by the modulus and recognise the point. **3. Products and quotients:** multiply or divide the moduli, add or subtract the arguments, then bring the argument into $(-\pi,\pi]$. **4. Powers:** De Moivre, or repeated squaring: $(1+i)^2=2i$, so $(1+i)^8=(2i)^4=16$. **5. Calculator:** ▶ Polar and ▶ Rect.

Question 4A.2** (8 marks) Solving polynomial equations

[Notes link: **4.5** Solving equations over C] Workbook p. 34

a. Solve $z^2+4z+13=0$ for $z\in C$. 2 marks

$(z+2)^2+9=0\Rightarrow z+2=\pm 3i\Rightarrow z=-2\pm 3i$. **M1 A1**

b. Given that $1-2i$ is a root of $P(z)=z^3-z^2+3z+5$, find the other roots of $P(z)=0$. 3 marks

P has real coefficients, hence by the conjugate root theorem $1+2i$ is also a root. $(z-(1-2i))(z-(1+2i))=z^2-2z+5$. **M1 M1**

$P(z)=(z^2-2z+5)(z+1)$ (by inspection or division), so the roots are $1\pm 2i$ and -1. **A1**

c. By letting $z=a+bi$, solve $z^2=-3+4i$ for $z\in C$. 3 marks

$(a+bi)^2=a^2-b^2+2abi=-3+4i$, so $a^2-b^2=-3$ and $ab=2$. Substituting $b=\frac{2}{a}$: $a^4+3a^2-4=0\Rightarrow(a^2+4)(a^2-1)=0\Rightarrow a=\pm 1$, $b=\pm 2$. $z=\pm(1+2i)$. **M1 M1 A1**

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$\text{cFactor}(z^3-z^2+3\cdot z+5,z)$	$(z+1)(z-1+2i)(z-1-2i)$
$\text{cSolve}(z^2=-3+4\cdot i,z)$	$z=-1-2i$ or $z=1+2i$

Ways to do it. **1. Completing the square:** $(z+2)^2=-9$. **2. Quadratic formula** with a negative discriminant. **3. Conjugate root theorem:** $1+2i$ is also a root, so z^2-2z+5 is a factor; divide, or get the real root from the sum of the roots. **4. Square roots by** $z=a+bi$**:** equate real and imaginary parts, and add $a^2+b^2=|-3+4i|=5$ to make it quick. **5. Calculator:** cSolve( ) (menu 3 C 1). **6. In the Notes:** techniques 3, 7 and 2 of the factorising table in Section 4.5.

Question 4A.3**** (9 marks) Roots of unity and trigonometry

[Notes link: **4.4** De Moivre's theorem] Workbook p. 35

Let $\omega=\text{cis}\left(\frac{2\pi}{5}\right)$.

a. Show that $1+\omega+\omega^2+\omega^3+\omega^4=0$. 2 marks

$\omega^5=\text{cis}(2\pi)=1$ and $\omega\neq 1$. Since $\omega^5-1=(\omega-1)(1+\omega+\omega^2+\omega^3+\omega^4)=0$ and $\omega-1\neq 0$, the second factor is 0. **M1 M1**

b. Hence show that $\cos\left(\frac{2\pi}{5}\right)+\cos\left(\frac{4\pi}{5}\right)=-\frac{1}{2}$. 3 marks

$\omega^4=\text{cis}\left(\frac{8\pi}{5}\right)=\text{cis}\left(-\frac{2\pi}{5}\right)=\overline{\omega}$ and $\omega^3=\overline{\omega^2}$. So $\omega+\omega^4=2\cos\frac{2\pi}{5}$ and $\omega^2+\omega^3=2\cos\frac{4\pi}{5}$. **M1 M1**
Part **a.** becomes $1+2\cos\frac{2\pi}{5}+2\cos\frac{4\pi}{5}=0$, so $\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=-\frac{1}{2}$. **M1**

c. Let $z=\text{cis}(\theta)$. Show that $z^n+z^{-n}=2\cos(n\theta)$ for $n\in N$. 1 mark

$z^n+z^{-n}=\text{cis}(n\theta)+\text{cis}(-n\theta)=2\cos(n\theta)$, since the imaginary parts cancel. **A1**

d. Solve $z^4-2z^2+4=0$ for $z\in C$, giving your answers in Cartesian form. 3 marks

A quadratic in z^2: $z^2=\dfrac{2\pm\sqrt{4-16}}{2}=1\pm\sqrt{3}i=2\,\text{cis}\left(\pm\frac{\pi}{3}\right)$. **M1 M1**

So $z=\pm\sqrt{2}\,\text{cis}\left(\pm\frac{\pi}{6}\right)$: $z=\pm\left(\dfrac{\sqrt{6}}{2}+\dfrac{\sqrt{2}}{2}i\right)$, $z=\pm\left(\dfrac{\sqrt{6}}{2}-\dfrac{\sqrt{2}}{2}i\right)$. **A1**

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$\text{cSolve}(z^4-2\cdot z^2+4=0,z)$
four roots $\pm\frac{\sqrt{6}}{2}\pm\frac{\sqrt{2}}{2}i$

$\cos\left(\dfrac{2\cdot\pi}{5}\right)+\cos\left(\dfrac{4\cdot\pi}{5}\right)$ $\quad -\frac{1}{2}$

The trick. The roots of unity add to zero. Take the real part of that statement and it becomes a fact about cosines. More generally, $z^n+z^{-n}=2\cos(n\theta)$ turns any symmetric expression in z into trigonometry.

Ways to do it. **1. Geometric series:** the sum is $\frac{\omega^5-1}{\omega-1}$ and $\omega^5=1$. **2. Factorise:** $z^5-1=(z-1)(z^4+z^3+z^2+z+1)$ and $\omega\neq 1$. **3. Vectors:** the five roots are evenly spaced, so they add to zero. **4. Real parts:** take the real part of the sum and pair ω^4 with ω, ω^3 with ω^2. **5.** $z^n+z^{-n}=2\cos(n\theta)$ from De Moivre with $\pm n$. **6. Quadratic in** z^2, then take square roots in polar form. **7. In the Notes:** technique 9 of the factorising table in Section 4.5.

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Question 4A.4*** (5 marks) Exact values from polar form

[Notes link: **4.3** Polar form; **4.5** Solving equations over C] Workbook p. 36

a. By writing $\operatorname{cis}\left(\frac{5\pi}{12}\right) = \operatorname{cis}\left(\frac{\pi}{4}\right)\operatorname{cis}\left(\frac{\pi}{6}\right)$, find the exact value of $\cos\left(\frac{5\pi}{12}\right)$. 2 marks

$\operatorname{cis}\frac{\pi}{4}\operatorname{cis}\frac{\pi}{6} = \left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) = \frac{\sqrt{6}-\sqrt{2}}{4}+\frac{\sqrt{6}+\sqrt{2}}{4}i$. Equating real parts: $\cos\frac{5\pi}{12} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$. **M1 A1**

b. Find the square roots of $1-\sqrt{3}i$ in Cartesian form. 3 marks

$1-\sqrt{3}i = 2\operatorname{cis}\left(-\frac{\pi}{3}\right)$, so the square roots are $\pm\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{6}\right) = \pm\left(\dfrac{\sqrt{6}}{2}-\dfrac{\sqrt{2}}{2}i\right)$. **M1 M1 A1**

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$\operatorname{cSolve}(z^2 = 1-\sqrt{3}\cdot i, z)$
$\Rightarrow z = \pm\left(\frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}i\right)$

Ways to do it. 1. Multiply in Cartesian form: expand $\operatorname{cis}\frac{\pi}{4}\operatorname{cis}\frac{\pi}{6}$ and read the real part. **2. Compound angle formula** for $\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$ directly. **3. Square roots in polar form:** $2\operatorname{cis}\left(-\frac{\pi}{3}\right)$ has roots $\pm\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{6}\right)$. **4. Square roots by** $z=a+bi$**:** $a^2-b^2=1$, $2ab=-\sqrt{3}$, $a^2+b^2=2$.

Section B – Technology active

Question 4B.1***** (11 marks) Roots on a line

[Notes link: **4.2** Cartesian form; **4.4** De Moivre's theorem; **4.5** Solving equations over C; **12.9** CAS Guide: Complex numbers] Workbook p. 37

a. Let $z \in C$ with $|z|=1$ and $z \neq 1$. Show that $\dfrac{z+1}{z-1}$ is purely imaginary (or zero). 3 marks

$\dfrac{z+1}{z-1} = \dfrac{(z+1)(\bar{z}-1)}{(z-1)(\bar{z}-1)} = \dfrac{z\bar{z}-z+\bar{z}-1}{|z-1|^2} = \dfrac{\bar{z}-z}{|z-1|^2}$, using $z\bar{z}=|z|^2=1$. **M1 M1**

With $z=x+yi$, $\bar{z}-z=-2yi$, so the quotient is $\dfrac{-2y}{|z-1|^2}i$: a real multiple of i (zero when $z=-1$). **A1**

b. Interpret the result of part **a.** geometrically, in terms of the points -1, 1 and z on an Argand diagram. 2 marks

$z+1$ is the vector from -1 to z and $z-1$ is the vector from 1 to z. A purely imaginary quotient means their arguments differ by $\pm\dfrac{\pi}{2}$, so the two vectors are perpendicular. That is the angle in a semicircle: -1 and 1 are the ends of a diameter of the unit circle, and z is on the circle. **M1 A1**

c. By letting $w = \dfrac{z+1}{z-1}$, solve $(z+1)^4 = (z-1)^4$ for $z \in C$. 3 marks

$z=1$ is not a solution, so divide: $w^4=1$, $w \in \{1, i, -1, -i\}$, and $w=1$ is impossible ($z+1=z-1$). **M1**

$w = \dfrac{z+1}{z-1} \Rightarrow z = \dfrac{w+1}{w-1}$: $\quad w=-1 \Rightarrow z=0$; $\ w=i \Rightarrow z=\dfrac{i+1}{i-1}=-i$; $\ w=-i \Rightarrow z=i$. **M1**

$z=0,\ i,\ -i$. (Check: $(z+1)^4-(z-1)^4 = 8z^3+8z = 8z(z^2+1)$.) **A1**

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d. Show that for any integer $n \geq 2$ the solutions of $(z+1)^n = (z-1)^n$ are $z = -i\cot\left(\dfrac{k\pi}{n}\right)$, $k = 1, 2, \ldots, n-1$, and state what these solutions have in common. 3 marks

$w^n = 1$ with $w \neq 1$: $w = \text{cis}(\theta)$, $\theta = \dfrac{2k\pi}{n}$, $k = 1, \ldots, n-1$. Then

$$z = \frac{\text{cis}\,\theta + 1}{\text{cis}\,\theta - 1} = \frac{2\cos\frac{\theta}{2}\,\text{cis}\,\frac{\theta}{2}}{2i\sin\frac{\theta}{2}\,\text{cis}\,\frac{\theta}{2}} = -i\cot\left(\frac{\theta}{2}\right) = -i\cot\left(\frac{k\pi}{n}\right),$$

using $\text{cis}\,\theta + 1 = 2\cos\frac{\theta}{2}\,\text{cis}\,\frac{\theta}{2}$ and $\text{cis}\,\theta - 1 = 2i\sin\frac{\theta}{2}\,\text{cis}\,\frac{\theta}{2}$. **M1 M1**

Every solution is purely imaginary (or zero): they all lie on the imaginary axis. There are $n-1$ of them because the z^n terms cancel. **M1**

The trick. An equation of the form (something)^n = (something else)^n is a roots-of-unity question in disguise: divide, call the ratio w, and solve $w^n = 1$. Part **a.** run backwards explains the answer: $|w| = 1$ forces z onto the imaginary axis, because z and w are related by the same formula in both directions.

Ways to do it. 1. Substitution $w = \frac{z+1}{z-1}$ and roots of unity. **2. Expand** for $n = 4$: the z^4 terms cancel, leaving a cubic that factorises. **3. Modulus argument:** $|z+1| = |z-1|$ says z is equidistant from -1 and 1, the perpendicular bisector, which is the imaginary axis. **4. Half-angle factorisations** of $\text{cis}\,\theta \pm 1$, or multiply top and bottom by the conjugate of $\text{cis}\,\theta - 1$. **5. Calculator:** $\text{cSolve}((z+1)^4 = (z-1)^4, z)$.

Question 4B.2*** (14 marks) Minor segments

[Notes link: **4.6** Sketching in the complex plane] Workbook p. 38

a. The line $\text{Re}(z) = 2$ cuts the circle $|z| = 4$ at the points P and Q. Find P and Q in Cartesian form, and show that the chord PQ subtends an angle of $\dfrac{2\pi}{3}$ at the centre of the circle. 2 marks

$x = 2$ in $x^2 + y^2 = 16$: $y = \pm 2\sqrt{3}$, so $P = 2 + 2\sqrt{3}i$ and $Q = 2 - 2\sqrt{3}i$. **M1**

The chord is 2 units from the centre and the radius is 4, so half the angle has $\cos = \dfrac{2}{4}$: half the angle is $\dfrac{\pi}{3}$ and the chord subtends $\dfrac{2\pi}{3}$. **A1**

b. Hence find the exact area of the minor segment of the circle cut off by the line, that is, the area of $\{z \in C : |z| \leq 4\} \cap \{z \in C : \text{Re}(z) \geq 2\}$. 2 marks

$$\text{Segment} = \text{sector} - \text{triangle} = \frac{1}{2}r^2(\theta - \sin\theta) = \frac{1}{2}(16)\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) = \frac{16\pi}{3} - 4\sqrt{3}$$ **M1 A1**

c. The ray $\text{Arg}(z) = \dfrac{\pi}{3}$ meets the circle $|z-2| = 2$ at the point A. Find A in Cartesian form. 2 marks

The ray is $y = \sqrt{3}x$ with $x > 0$. In $(x-2)^2 + y^2 = 4$: $(x-2)^2 + 3x^2 = 4 \Rightarrow 4x^2 - 4x = 0 \Rightarrow x = 1$. $A = 1 + \sqrt{3}i$ **M1 A1**

d. The circle $|z-2| = 2$ passes through the origin O. Find the exact area of the minor segment of this circle cut off by the chord OA. 3 marks

From the centre $(2, 0)$, the vectors to O and to A are $-2\underset{\sim}{i}$ and $-\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j}$. **M1 M1**

$\cos\theta = \dfrac{2}{2\times 2} = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{3}$. Area $= \dfrac{1}{2}(4)\left(\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{2}\right) = \dfrac{2\pi}{3} - \sqrt{3}$ **A1**

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e. Sketch $\{z \in C : |z| \leq 2\sqrt{2}\} \cap \{z \in C : |z-2-2i| \leq |z|\}$. 2 marks

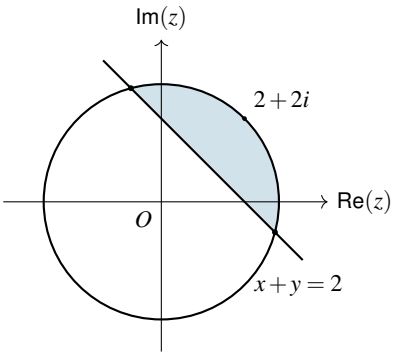
A disc of radius $2\sqrt{2}$ centred at the origin. $|z-2-2i| \leq |z|$ is the set of points at least as close to $2+2i$ as to 0: the side of the perpendicular bisector $x+y=2$ that contains $2+2i$. The region is the minor segment of the disc beyond that line. **A1 A1**

f. Find the exact area of the region sketched in part **e.** 3 marks

The line $x+y=2$ is $\dfrac{2}{\sqrt{2}} = \sqrt{2}$ from the origin, and the radius is $2\sqrt{2}$, so half the angle has $\cos = \dfrac{1}{2}$ and the chord subtends $\dfrac{2\pi}{3}$. **M1**

Area $= \dfrac{1}{2}(8)\left(\dfrac{2\pi}{3} - \dfrac{\sqrt{3}}{2}\right) = \dfrac{8\pi}{3} - 2\sqrt{3}$ **M1**

(The chord ends are $(1+\sqrt{3}, 1-\sqrt{3})$ and $(1-\sqrt{3}, 1+\sqrt{3})$.) **A1**



e.

the region — boundary included - - - boundary not included ○ point not included

The trick. Every minor segment question is the same three steps: find where the line or ray meets the circle, find the angle the chord subtends *at the centre of the circle* (a right-angled triangle with the distance from the centre to the chord, or a dot product of the two radius vectors), then use $\frac{1}{2}r^2(\theta - \sin\theta)$ with θ in radians. The mistake is measuring the angle at the origin when the origin is not the centre.

Ways to do it. **1.** Sector minus triangle, $\frac{1}{2}r^2(\theta - \sin\theta)$. **2.** Angle at the centre from $\cos\left(\frac{\theta}{2}\right) = \dfrac{\text{distance from centre to chord}}{r}$, or from the dot product of the two radius vectors. **3.** Integration on the calculator as a check: for part **b.**, $2\displaystyle\int_2^4 \sqrt{16-x^2}\,dx = 9.827$. **4.** Major segment = area of the circle minus the minor segment. **5.** The other questions with segments are Section A Question 13 and Section B Questions 5 and 7.

Section C – Multiple choice (technology active)

Question 4C.1*** (1 mark) Purely imaginary powers

[Notes link: **4.4** De Moivre's theorem] Workbook p. 40

Let $z = b - bi$, where $b > 0$. Then z^n, $n \in Z$, is purely imaginary exactly when

A. $n = 4k, k \in Z$ **B.** $n = 2k+1, k \in Z$ **C.** $n = 4k+2, k \in Z$ **D.** $n = 8k+2, k \in Z$

Solution. $z = \sqrt{2}b\,\text{cis}\left(-\frac{\pi}{4}\right)$, so z^n has argument $-\frac{n\pi}{4}$, which must be an odd multiple of $\frac{\pi}{2}$: $n = 2, 6, 10, \ldots$ (and negatives). **C**

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Question 4C.2** (1 mark) Intersecting rays

[Notes link: **4.6** Sketching in the complex plane] Workbook p. 40

The rays $\text{Arg}(z) = \dfrac{\pi}{3}$ and $\text{Arg}(z-1) = \dfrac{2\pi}{3}$ intersect at

A. $\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$ **B.** $1 + \sqrt{3}i$ **C.** $\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$ **D.** $\sqrt{3}i$

Solution. $y = \sqrt{3}x$ $(x > 0)$ and $y = -\sqrt{3}(x-1)$ $(x < 1)$ meet at $x = \frac{1}{2}$. **A**

Question 4C.3**** (1 mark) An equation with a conjugate

[Notes link: **4.3** Polar form; **4.5** Solving equations over C] Workbook p. 40

The number of distinct solutions of $z^3 = 4\bar{z}$, where $z \in C$, is

A. 3 **B.** 4 **C.** 5 **D.** 6

Solution. $z = 0$ is one solution. Otherwise take the modulus of both sides: $|z|^3 = 4|z|$, so $|z| = 2$. Multiply both sides by z: $z^4 = 4z\bar{z} = 4|z|^2 = 16$, which has four solutions, ± 2 and $\pm 2i$, all with modulus 2. That makes $1 + 4 = 5$. The conjugate means this is not a polynomial equation in z, so "a cubic has three solutions" does not apply. **C**

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5 Differential Calculus and Sketching

Section A – Technology free

Question 5A.1* (6 marks) Derivatives

[Notes link: **5.1** Differentiation formulae] Workbook p. 43

Differentiate each of the following with respect to x.

a. $\sin^{-1}(3x)$ 1 mark

$\dfrac{3}{\sqrt{1-9x^2}}$ **A1**

b. $\tan^{-1}\left(\dfrac{x}{2}\right)$ 1 mark

$\dfrac{1/2}{1+\frac{x^2}{4}} = \dfrac{2}{4+x^2}$ **A1**

c. $x\cos^{-1}(x)$ 2 marks

Product rule: $\cos^{-1}(x) - \dfrac{x}{\sqrt{1-x^2}}$ **M1 A1**

d. $\sec(2x)$ 1 mark

$2\sec(2x)\tan(2x)$ **A1**

e. $\cot(x)$ 1 mark

$-\operatorname{cosec}^2(x)$ **A1**

Ways to do it. 1. Chain rule with the inverse circular derivatives: $\frac{d}{dx}\sin^{-1}(3x) = \frac{3}{\sqrt{1-9x^2}}$; the inside's derivative goes on top. **2. Formula sheet form:** $\tan^{-1}\frac{x}{a}$ differentiates to $\frac{a}{a^2+x^2}$. **3. Product rule** for $x\cos^{-1}(x)$. **4. Rewrite first:** $\sec 2x = (\cos 2x)^{-1}$ and $\cot x = \frac{\cos x}{\sin x}$, if you forget the reciprocal derivatives. **5. Calculator** derivative as a check (menu 4 1).

Question 5A.2** (6 marks) Second derivatives and inflection

[Notes link: **5.2** The second derivative] Workbook p. 44

a. Let $f(x) = \tan^{-1}(x)$. Find $f''(x)$ and hence find the coordinates of the point of inflection of the graph of f. 3 marks

$f'(x) = \dfrac{1}{1+x^2}$, $f''(x) = -\dfrac{2x}{(1+x^2)^2}$. $f'' = 0$ at $x = 0$, and f'' changes sign there (positive for $x < 0$, negative for $x > 0$). The point of inflection is $(0,0)$. **M1 M1 A1**

b. Let $g(x) = x^4 - 4x^3$. Find the coordinates of the points of inflection of the graph of g, and state whether each is a stationary point of inflection. 3 marks

$g'(x) = 4x^3 - 12x^2$, $g''(x) = 12x^2 - 24x = 12x(x-2)$. g'' changes sign at $x = 0$ and at $x = 2$. **M1 M1**
$(0,0)$: $g'(0) = 0$, so this is a **stationary** point of inflection. $(2,-16)$: $g'(2) = -16 \neq 0$, so it is a non-stationary point of inflection. **A1**

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Ways to do it. 1. $f''(x)=0$ and a sign change: solve, then test f'' either side. **2. Repeated-root check:** $g''(x)=12x(x-2)$ has two single roots, so both are inflections; a squared factor would not be. **3. Stationary point of inflection** if f' is also zero there ($x=0$ for g). **4. Graph** f'' on the calculator and look for where it crosses the axis.

Question 5A.3**** (10 marks) Proving with derivatives

[Notes link: **5.1** Differentiation formulae] Workbook p. 44

a. Let $f(x)=\sin^{-1}(x)+\cos^{-1}(x)$ for $x\in(-1,1)$. Show that $f'(x)=0$, and hence find $f(x)$. 2 marks

$f'(x)=\dfrac{1}{\sqrt{1-x^2}}-\dfrac{1}{\sqrt{1-x^2}}=0$, so f is constant: $f(x)=f(0)=0+\dfrac{\pi}{2}=\dfrac{\pi}{2}$. **M1 A1**

b. Let $y=\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$, $x<1$. Show that $\dfrac{dy}{dx}=\dfrac{1}{1+x^2}$. 3 marks

Let $u=\dfrac{1+x}{1-x}$, so $\dfrac{du}{dx}=\dfrac{(1-x)+(1+x)}{(1-x)^2}=\dfrac{2}{(1-x)^2}$. **M1 M1**

$\dfrac{dy}{dx}=\dfrac{1}{1+u^2}\cdot\dfrac{du}{dx}=\dfrac{(1-x)^2}{(1-x)^2+(1+x)^2}\cdot\dfrac{2}{(1-x)^2}=\dfrac{2}{2+2x^2}=\dfrac{1}{1+x^2}$. **A1**

c. Hence express $\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$ in terms of $\tan^{-1}(x)$ for $x<1$. 2 marks

Both functions have the same derivative on $x<1$, so they differ by a constant: $\tan^{-1}\left(\frac{1+x}{1-x}\right)=\tan^{-1}(x)+c$. At $x=0$: $\frac{\pi}{4}=0+c$. So $\tan^{-1}\left(\dfrac{1+x}{1-x}\right)=\tan^{-1}(x)+\dfrac{\pi}{4}$. **M1 A1**

d. For the circle $x^2+y^2=25$, show that $\dfrac{d^2y}{dx^2}=-\dfrac{25}{y^3}$. 3 marks

$2x+2y\dfrac{dy}{dx}=0\Rightarrow\dfrac{dy}{dx}=-\dfrac{x}{y}$. Then $\dfrac{d^2y}{dx^2}=-\dfrac{y-x\frac{dy}{dx}}{y^2}=-\dfrac{y+\frac{x^2}{y}}{y^2}=-\dfrac{x^2+y^2}{y^3}=-\dfrac{25}{y^3}$. **M1 M1 A1**

TI-Nspire | check

$\text{impDif}(x^2+y^2=25,x,y,2)$

second derivative: simplifies to $-\frac{x^2+y^2}{y^3}$

The trick. If a derivative is zero, the function is a constant; if two functions have the same derivative, they differ by a constant. Either way, calculus does the hard part and one easy substitution (usually $x=0$) finds the constant.

Ways to do it. 1. Zero derivative means constant: then find the constant with an easy value such as $x=0$. **2. Triangle argument:** $\sin^{-1}x$ and $\cos^{-1}x$ are the two acute angles of a right-angled triangle, so they add to $\frac{\pi}{2}$. **3. Chain and quotient rules,** then simplify $(1-x)^2+(1+x)^2=2(1+x^2)$. **4. Same derivative, so they differ by a constant:** $\tan^{-1}\left(\frac{1+x}{1-x}\right)=\tan^{-1}x+\frac{\pi}{4}$, checked at $x=0$. **5. Compound angle check:** $\tan\left(A+\frac{\pi}{4}\right)=\frac{1+\tan A}{1-\tan A}$.

Question 5A.4*** (10 marks) Circles, ellipses and hyperbolas

[Notes link: **5.10** Circles, ellipses, hyperbolas and parametric curves] Workbook p. 46

a. Sketch $\dfrac{(x-1)^2}{9}+\dfrac{(y+2)^2}{4}=1$, labelling the centre and the endpoints of the axes. 3 marks

Centre $(1,-2)$; horizontal semi-axis 3, vertical semi-axis 2. Endpoints $(-2,-2)$, $(4,-2)$, $(1,0)$, $(1,-4)$.

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A1 A1 A1

b. Sketch $\frac{x^2}{4}-\frac{y^2}{9}=1$, labelling the vertices and the asymptotes. 3 marks

Vertices $(\pm 2,0)$; asymptotes $y=\pm\frac{3}{2}x$. Two branches, opening left and right. **A1 A1 A1**

c. Find the Cartesian equation of the curve $x=\sec(t)$, $y=2\tan(t)$, $t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, and state which part of the graph it traces. 2 marks

$\sec^2 t-\tan^2 t=1\Rightarrow x^2-\frac{y^2}{4}=1$. On $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $x=\sec t\geq 1$, so it is the right branch only (all y-values). **M1 A1**

d. Find the centre and radius of the circle $x^2+y^2-6x+4y-12=0$. 2 marks

$(x-3)^2-9+(y+2)^2-4-12=0\Rightarrow(x-3)^2+(y+2)^2=25$. Centre $(3,-2)$, radius 5. **M1 A1**

TI-Nspire | check

$\text{completeSquare}(x^2+y^2-6\cdot x+4\cdot y-12=0,x,y)$ $\qquad (x-3)^2+(y+2)^2=25$

Ways to do it. 1. Standard form: read the centre and the semi-axes; complete the square first if it is expanded. **2. Hyperbola:** vertices $(\pm a,0)$ and asymptotes $y=\pm\frac{b}{a}x$; draw the box and its diagonals. **3. Parametric to Cartesian:** isolate the trig functions and use $\cos^2+\sin^2=1$ or $\sec^2-\tan^2=1$. **4. Calculator:** menu 3 2 Relation, or parametric mode, to check the sketch.

Section B – Technology active

Question 5B.1*** (7 marks) The folium

[Notes link: **5.3** Implicit differentiation; **12.10** CAS Guide: Differential calculus] Workbook p. 47

A curve has equation $x^3+y^3=6xy$.

a. Find $\frac{dy}{dx}$ in terms of x and y. 2 marks

$3x^2+3y^2\frac{dy}{dx}=6y+6x\frac{dy}{dx}\Rightarrow\frac{dy}{dx}=\frac{2y-x^2}{y^2-2x}$ **M1 A1**

b. Find the equation of the tangent to the curve at $(3,3)$. 2 marks

At $(3,3)$: $\frac{dy}{dx}=\frac{6-9}{9-6}=-1$, so $y=-x+6$. **M1 A1**

c. Find the coordinates of the point, other than the origin, where the tangent to the curve is horizontal. Give your answer correct to three decimal places. 3 marks

Horizontal when $2y=x^2$. Solve with the curve: $(2.520,\ 3.175)$. **M1 M1 A1**

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$\text{impDif}(x^3+y^3=6\cdot x\cdot y,x,y)$ $\qquad \frac{-(x^2-2y)}{y^2-2x}$

$\text{solve}(x^3+y^3=6\cdot x\cdot y \text{ and } 2\cdot y=x^2,\{x,y\})$

$x=0,\ y=0$ or $x=2.51984\ldots,\ y=3.17480\ldots$ (exactly $x=2^{4/3}$, $y=2^{5/3}$)

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Ways to do it. 1. Implicit differentiation, with the product rule on $6xy$. **2. Symmetry:** the curve is unchanged when x and y swap, so the tangent at $(3,3)$ has gradient -1. **3. Horizontal tangent:** numerator zero gives $y=\frac{x^2}{2}$; substitute into the curve. **4. Calculator:** $\text{impDif}(\ )$, then solve the condition with the curve.

Question 5B.2***** (11 marks) Around the corner

[Notes link: **5.1** Differentiation formulae; **5.5** Sketching rational functions; **12.10** CAS Guide: Differential calculus]

Workbook p. 48

A corridor 8 m wide meets a corridor 1 m wide at a right angle. A straight pole is carried horizontally around the corner. At the tightest moment the pole touches the inside corner and both outside walls, making an angle θ with the outside wall of the wide corridor, where $0<\theta<\dfrac{\pi}{2}$.

a. Show that the length of a pole that just touches all three is $L(\theta)=8\sec(\theta)+\operatorname{cosec}(\theta)$. 2 marks

The inside corner splits the pole into two pieces. The piece that spans the wide corridor is the hypotenuse of a right-angled triangle whose side *across* the corridor is 8 and is adjacent to θ: length $\dfrac{8}{\cos\theta}$. The piece that spans the narrow corridor has the width 1 opposite θ: length $\dfrac{1}{\sin\theta}$. So $L=8\sec\theta+\operatorname{cosec}\theta$. **M1 M1**

b. Find $L'(\theta)$. 2 marks

$L'(\theta)=8\sec\theta\tan\theta-\operatorname{cosec}\theta\cot\theta$ **M1 A1**

c. Show that $L'(\theta)=0$ when $\tan(\theta)=\dfrac{1}{2}$, and find the exact value of L there. 3 marks

$L'=0\Rightarrow\dfrac{8\sin\theta}{\cos^2\theta}=\dfrac{\cos\theta}{\sin^2\theta}\Rightarrow\tan^3\theta=\dfrac{1}{8}\Rightarrow\tan\theta=\dfrac{1}{2}$. **M1 M1**

Then $\sin\theta=\dfrac{1}{\sqrt{5}}$, $\cos\theta=\dfrac{2}{\sqrt{5}}$: $L=8\times\dfrac{\sqrt{5}}{2}+\sqrt{5}=5\sqrt{5}\approx11.18\,\text{m}$. **M1**

d. Is this value a maximum or a minimum of L? Explain what it means for the longest pole that can be carried around the corner. 2 marks

$L\to\infty$ as $\theta\to0$ and as $\theta\to\dfrac{\pi}{2}$, so the single stationary point is a **minimum**. $L(\theta)$ is the longest pole that fits at angle θ, and the pole must get through *every* angle on the way around, so the tightest angle decides: the longest pole that can be carried around is the *minimum* of L, $5\sqrt{5}\,\text{m}$. **M1 A1**

e. The narrow corridor is widened to w metres so that a 13 m pole just fits. Find w, correct to two decimal places. 2 marks

With widths 8 and w, the same working gives $\tan^3\theta=\dfrac{w}{8}$ and a minimum of $\left(8^{\frac{2}{3}}+w^{\frac{2}{3}}\right)^{\frac{3}{2}}$. **M1**

$\left(4+w^{\frac{2}{3}}\right)^{\frac{3}{2}}=13\Rightarrow w=\left(13^{\frac{2}{3}}-4\right)^{\frac{3}{2}}=1.89\,\text{m}$. **A1**

TI-Nspire | menu 4 7: fMin, and solve

$\text{fMin}\left(\dfrac{8}{\cos(x)}+\dfrac{1}{\sin(x)},x\right)\mid 0<x<\dfrac{\pi}{2}$	$x=\tan^{-1}\left(\frac{1}{2}\right)$
$\dfrac{8}{\cos(x)}+\dfrac{1}{\sin(x)}\mid x=\tan^{-1}\left(\dfrac{1}{2}\right)$	$5\sqrt{5}$
$\text{nSolve}\left(\left(4+w^{\frac{2}{3}}\right)^{\frac{3}{2}}=13,w\right)$	1.89023

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Watch out. The answer to "the longest pole" is a *minimum*. If you expect a maximum and test the nature carelessly you will talk yourself into the wrong conclusion.

The trick. Let the pole touch everything at once and write its length as a function of the angle. The pole that gets stuck is the one that is too long at the worst angle, so you minimise.

Ways to do it. **1. Differentiate** $L(\theta)$ with the reciprocal function derivatives. **2. Sine and cosine form:** $L=\frac{8}{\cos\theta}+\frac{1}{\sin\theta}$ and the quotient rule. **3. Coordinates:** a line through the corner point $(8,1)$ with intercepts a and b; minimise $\sqrt{a^2+b^2}$ subject to $\frac{8}{a}+\frac{1}{b}=1$. **4. General result:** $\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$ for widths a and b. **5. Calculator:** fMin, or graph L and use Analyze Graph.

Section C – Multiple choice (technology active)

Question 5C.1*** (1 mark) No turning points

[Notes link: **5.5** Sketching rational functions] Workbook p. 50

The graph of $y=\dfrac{x^2+px+1}{x-1}$, where $p\in R$, has no turning points exactly when

A. $p<2$ **B.** $p\leq -2$ **C.** $p>2$ **D.** $p>-2$

Solution. $y=x+p+1+\frac{p+2}{x-1}$, so $\frac{dy}{dx}=1-\frac{p+2}{(x-1)^2}$, which is never 0 when $p+2\leq 0$. ($p=-2$ gives a line with a hole.) **B**

Question 5C.2*** (1 mark) Inflection points with a parameter

[Notes link: **5.2** The second derivative] Workbook p. 50

The graph of $f(x)=ax^4+2x^3+3x^2$, $a\in R$, has two points of inflection when

A. $a<\dfrac{1}{2}$ **B.** $a<\dfrac{1}{2}$ and $a\neq 0$ **C.** $a>\dfrac{1}{2}$ **D.** $a\neq 0$

Solution. $f''(x)=12ax^2+12x+6$ needs two distinct roots: $144-288a>0$. If $a=0$, f'' is linear (only one inflection point). **B**

Question 5C.3** (1 mark) Reciprocal of a quadratic

[Notes link: **5.5** Sketching rational functions] Workbook p. 50

The graph of $y=\dfrac{1}{x^2-4x+3}$ has

A. a local minimum at $(2,-1)$ **B.** a local maximum at $(2,1)$

C. a local maximum at $(2,-1)$ **D.** a single vertical asymptote $x=2$

Solution. The parabola x^2-4x+3 has a minimum -1 at $x=2$, so its reciprocal has a local maximum -1 there. Asymptotes $x=1$, $x=3$, $y=0$. **C**

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6 Integration Techniques

Section A – Technology free

Question 6A.1* (6 marks) Standard integrals

[Notes link: **6.1** Integration formulae and definite integrals] Workbook p. 53

Find each of the following.

a. $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$ 1 mark

$\sin^{-1}\left(\frac{x}{3}\right)+c$ **A1**

b. $\displaystyle\int \frac{4}{4+x^2}\,dx$ 1 mark

$2\displaystyle\int \frac{2}{2^2+x^2}\,dx = 2\tan^{-1}\left(\frac{x}{2}\right)+c$ **A1**

c. $\displaystyle\int \sec^2(3x)\,dx$ 1 mark

$\frac{1}{3}\tan(3x)+c$ **A1**

d. $\displaystyle\int \frac{-2}{\sqrt{1-4x^2}}\,dx$ 1 mark

$-\displaystyle\int \frac{2}{\sqrt{1-(2x)^2}}\,dx = -\sin^{-1}(2x)+c$ (or $\cos^{-1}(2x)+c$) **A1**

e. $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$ 1 mark

$\left[\sin^{-1}x\right]_0^{1/2} = \frac{\pi}{6}$ **A1**

f. $\displaystyle\int_0^{\sqrt{3}} \frac{1}{1+x^2}\,dx$ 1 mark

$\left[\tan^{-1}x\right]_0^{\sqrt{3}} = \frac{\pi}{3}$ **A1**

Ways to do it. 1. Formula sheet forms: $\frac{1}{\sqrt{a^2-x^2}} \to \sin^{-1}\frac{x}{a}$ and $\frac{a}{a^2+x^2} \to \tan^{-1}\frac{x}{a}$; fix the constant on top first. **2. Linear inside:** divide by the coefficient of x (so $\sec^2(3x) \to \frac{1}{3}\tan(3x)$, and write $4x^2=(2x)^2$). **3. Differentiate your answer** to check. **4. Calculator** integral (menu 4 3).

Question 6A.2** (6 marks) Linear substitution

[Notes link: **6.4** Linear substitution] Workbook p. 54

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a. Using the substitution $u = x-1$, find $\int x\sqrt{x-1}\,dx$. 3 marks

$x = u+1$, $dx = du$: $\int (u+1)u^{\frac{1}{2}}\,du = \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + c$ **M1 M1 A1**

b. Using a suitable substitution, evaluate $\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$. 3 marks

$u = x+1$, terminals $1 \to 4$: $\int_1^4 \frac{u-1}{\sqrt{u}}\,du = \left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^4 = \left(\frac{16}{3} - 4\right) - \left(\frac{2}{3} - 2\right) = \frac{8}{3}$ **M1 M1 A1**

Ways to do it. 1. Linear substitution: $u = x-1$, so $x = u+1$; expand and integrate powers of u. **2. Or** $u = \sqrt{x-1}$, which removes the root completely. **3. By parts** with $u = x$. **4. Rewrite the top:** $\frac{x}{\sqrt{x+1}} = \sqrt{x+1} - \frac{1}{\sqrt{x+1}}$, no substitution needed.

Question 6A.3*** (10 marks) Partial fractions

[Notes link: **6.5** Partial fractions] Workbook p. 54

a. Find $\int \frac{5}{(x-2)(x+3)}\,dx$. 3 marks

$\frac{5}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$: $x = 2$ gives $A = 1$, $x = -3$ gives $B = -1$. **M1 M1**

$\log_e|x-2| - \log_e|x+3| + c = \log_e\left|\frac{x-2}{x+3}\right| + c$ **A1**

b. Express $\frac{3x+1}{(x-1)(x+1)^2}$ in partial fractions. 3 marks

$3x+1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$. $x = 1$: $A = 1$. $x = -1$: $-2 = -2C \Rightarrow C = 1$. x^2: $0 = A + B \Rightarrow B = -1$. **M1 M1**

$\frac{1}{x-1} - \frac{1}{x+1} + \frac{1}{(x+1)^2}$ **A1**

c. Hence evaluate $\int_2^3 \frac{3x+1}{(x-1)(x+1)^2}\,dx$. 2 marks

$\left[\log_e(x-1) - \log_e(x+1) - \frac{1}{x+1}\right]_2^3 = \left(\log_e 2 - \log_e 4 - \frac{1}{4}\right) - \left(0 - \log_e 3 - \frac{1}{3}\right) = \log_e\left(\frac{3}{2}\right) + \frac{1}{12}$ **M1 A1**

d. Find $\int \frac{2x+3}{x^2+1}\,dx$. 2 marks

$\int \left(\frac{2x}{x^2+1} + \frac{3}{x^2+1}\right)dx = \log_e(x^2+1) + 3\tan^{-1}(x) + c$ **M1 A1**

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$\text{expand}\left(\frac{3\cdot x+1}{(x-1)\cdot(x+1)^2}, x\right)$	$\frac{1}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x-1}$
$\int_2^3 \frac{3\cdot x+1}{(x-1)\cdot(x+1)^2}\,dx$	$\ln\left(\frac{3}{2}\right) + \frac{1}{12}$

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Ways to do it. 1. Cover-up or substitution of the roots to find A and B. **2. Compare coefficients** as an alternative. **3. Repeated factor:** $\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$. **4. Split the top:** $\frac{2x}{x^2+1}+\frac{3}{x^2+1}$ gives a log and an inverse tan. **5. Calculator:** $\text{expand}(\ )$ gives partial fractions.

Question 6A.4**** (10 marks) Roots that hide a modulus

[Notes link: **6.7** Integrating a modulus; **6.3** Powers of sine and cosine; **7.4** Lengths of curves] Workbook p. 55

a. Show that $1+\sin(2x)=\big(\sin(x)+\cos(x)\big)^2$. 1 mark

$\big(\sin(x)+\cos(x)\big)^2=\sin^2(x)+2\sin(x)\cos(x)+\cos^2(x)=1+\sin(2x)$ **M1**

b. Hence evaluate $\int_0^{\pi}\sqrt{1+\sin(2x)}\,dx$. 4 marks

$\sqrt{1+\sin(2x)}=\sqrt{\big(\sin(x)+\cos(x)\big)^2}=|\sin(x)+\cos(x)|$. **M1**

$\sin(x)+\cos(x)=0$ when $\tan(x)=-1$, at $x=\frac{3\pi}{4}$. It is positive on $\left[0,\frac{3\pi}{4}\right)$ and negative on $\left(\frac{3\pi}{4},\pi\right]$. **M1**

$\int_0^{\frac{3\pi}{4}}\big(\sin(x)+\cos(x)\big)dx-\int_{\frac{3\pi}{4}}^{\pi}\big(\sin(x)+\cos(x)\big)dx=\Big[-\cos(x)+\sin(x)\Big]_0^{\frac{3\pi}{4}}-\Big[-\cos(x)+\sin(x)\Big]_{\frac{3\pi}{4}}^{\pi}$ **M1**

$=\Big(\sqrt{2}-(-1)\Big)-\Big(1-\sqrt{2}\Big)=2\sqrt{2}$ **A1**

c. Evaluate $\int_0^{2\pi}\sqrt{1+\cos(x)}\,dx$. 3 marks

$1+\cos(x)=2\cos^2\left(\frac{x}{2}\right)$, so the integrand is $\sqrt{2}\left|\cos\left(\frac{x}{2}\right)\right|$. $\cos\left(\frac{x}{2}\right)\geq 0$ on $[0,\pi]$ and ≤ 0 on $[\pi,2\pi]$. **M1 M1**

$\sqrt{2}\int_0^{\pi}\cos\left(\frac{x}{2}\right)dx-\sqrt{2}\int_{\pi}^{2\pi}\cos\left(\frac{x}{2}\right)dx=\sqrt{2}\Big[2\sin\left(\frac{x}{2}\right)\Big]_0^{\pi}-\sqrt{2}\Big[2\sin\left(\frac{x}{2}\right)\Big]_{\pi}^{2\pi}=2\sqrt{2}+2\sqrt{2}=4\sqrt{2}$ **A1**

d. A student writes $\int_0^{\pi}\sqrt{1+\sin(2x)}\,dx=\Big[-\cos(x)+\sin(x)\Big]_0^{\pi}=2$. Explain the mistake, and state the largest value of b for which this method gives the correct value of $\int_0^{b}\sqrt{1+\sin(2x)}\,dx$. 2 marks

The student replaced $\sqrt{(\ldots)^2}$ with the bracket instead of its modulus. On $\left(\frac{3\pi}{4},\pi\right]$ the bracket is negative, so that part of the area was subtracted instead of added. **M1**

The method is correct as long as $\sin(x)+\cos(x)\geq 0$ on the whole interval: the largest value is $b=\frac{3\pi}{4}$. **A1**

The trick. A square root of a perfect square is a modulus: $\sqrt{u^2}=|u|$. The double-angle formulas turn $1\pm\sin(2x)$ and $1\pm\cos(2x)$ into perfect squares, so the root disappears, but a modulus is left behind and it has to be split where the inside changes sign.

Watch out. The wrong answer in **d** is a believable number, 2. Nothing about it looks wrong. The check is to ask whether the bracket changes sign between the terminals *before* integrating. A second check: the integrand is never negative and is about 1 on average, so the answer should be near $\pi\approx 3.1$, and $2\sqrt{2}\approx 2.83$ is.

Ways to do it. 1. Perfect square, then split the modulus (Notes 7.4 and 6.7). **2. Write it as one wave:** $\sin(x)+\cos(x)=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$, which makes the sign change at $\frac{3\pi}{4}$ easy to see. **3. Symmetry for c:** $\left|\cos\left(\frac{x}{2}\right)\right|$ is symmetric about $x=\pi$, so the answer is $2\times\sqrt{2}\int_0^{\pi}\cos\left(\frac{x}{2}\right)dx$. **4. Calculator check:** the definite integral template handles the modulus for you; use it to check, since Exam 1 will not.

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Section B – Technology active

Question 6B.1*** (6 marks) Partial fractions on the CAS

[Notes link: **6.5** Partial fractions] Workbook p. 57

Let $f(x) = \dfrac{4}{x^2(x+2)}$ for $x > 0$.

a. Express $f(x)$ in partial fractions. 2 marks

$$\frac{4}{x^2(x+2)} = -\frac{1}{x} + \frac{2}{x^2} + \frac{1}{x+2}$$ **M1 A1**

b. Show that $\displaystyle\int_1^2 f(x)\,dx = 1 + \log_e\left(\frac{2}{3}\right)$. 2 marks

$$\left[-\log_e x - \frac{2}{x} + \log_e(x+2)\right]_1^2 = (-\log_e 2 - 1 + \log_e 4) - (-2 + \log_e 3) = 1 + \log_e 2 - \log_e 3 = 1 + \log_e\left(\frac{2}{3}\right)$$ **M1 M1**

c. Find the value of $k > 1$ such that $\displaystyle\int_1^k f(x)\,dx = \frac{1}{2}$, correct to three decimal places. 2 marks

Solve $\displaystyle\int_1^k f(x)\,dx = \frac{1}{2}$ for $k > 1$: $k \approx 1.693$. **M1 A1**

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$\text{expand}\left(\dfrac{4}{x^2\cdot(x+2)}, x\right)$	$\frac{1}{x+2} + \frac{2}{x^2} - \frac{1}{x}$
$\displaystyle\int_1^2 \frac{4}{x^2\cdot(x+2)}\,dx$	$\ln\left(\frac{2}{3}\right) + 1$
$\text{nSolve}\left(\displaystyle\int_1^k \frac{4}{x^2\cdot(x+2)}\,dx = \frac{1}{2}, k\right) \mid k > 1$	$1.69303\ldots$

Ways to do it. 1. Partial fractions by hand: $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$. **2. Calculator:** expand( ). **3. Show:** integrate each term and combine the logs. **4. Unknown terminal:** solve( ) or nSolve( ) on the integral equation with $k > 1$.

Question 6B.2***** (12 marks) Two reduction formulas

[Notes link: **6.6** Integration by parts; **6.3** Powers of sine and cosine] Workbook p. 58

Let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n(x)\,dx$ and $T_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n(x)\,dx$, where n is a non-negative integer.

a. Use integration by parts to show that $I_n = \dfrac{n-1}{n} I_{n-2}$ for $n \geq 2$. 4 marks

Write $\sin^n(x) = \sin^{n-1}(x)\sin(x)$, with $u = \sin^{n-1}(x)$ and $\dfrac{dv}{dx} = \sin(x)$:

$$I_n = \left[-\sin^{n-1}(x)\cos(x)\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)\cos^2(x)\,dx$$

$$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)\left(1 - \sin^2(x)\right)dx = (n-1)I_{n-2} - (n-1)I_n.$$

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So $nI_n = (n-1)I_{n-2}$ and $I_n = \dfrac{n-1}{n}I_{n-2}$. **M1 M1 M1 M1**

b. Hence find the exact values of I_5 and I_6. 3 marks

$I_1 = 1$: $I_3 = \dfrac{2}{3}$, $I_5 = \dfrac{4}{5}\times\dfrac{2}{3} = \dfrac{8}{15}$. $\quad I_0 = \dfrac{\pi}{2}$: $I_2 = \dfrac{\pi}{4}$, $I_4 = \dfrac{3\pi}{16}$, $I_6 = \dfrac{5}{6}\times\dfrac{3\pi}{16} = \dfrac{5\pi}{32}$. **M1 M1 A1**

c. Show that $T_n + T_{n-2} = \dfrac{1}{n-1}$ for $n \geq 2$. 3 marks

$$T_n + T_{n-2} = \int_0^{\frac{\pi}{4}} \tan^{n-2}(x)\left(\tan^2(x)+1\right)dx = \int_0^{\frac{\pi}{4}} \tan^{n-2}(x)\sec^2(x)\,dx = \left[\frac{\tan^{n-1}(x)}{n-1}\right]_0^{\frac{\pi}{4}} = \frac{1}{n-1}$$ **M1 M1 M1**

d. Hence find the exact value of T_4, and check it on your calculator. 2 marks

$T_0 = \dfrac{\pi}{4}$, $T_2 = 1 - \dfrac{\pi}{4}$, $T_4 = \dfrac{1}{3} - T_2 = \dfrac{\pi}{4} - \dfrac{2}{3} \approx 0.1187$ **M1 A1**

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$\int_0^{\pi/4} \cdot(\tan(x))^4 \cdot dx$ $\quad \frac{3\pi-8}{12}$

The trick. The integral you want will reappear on the right-hand side. That is the goal, not a problem: collect the I_n terms on one side. For the tangent formula no integration by parts is needed, just $\tan^2+1 = \sec^2$.

Ways to do it. **1.** Integration by parts, splitting off one factor of $\sin(x)$, then $\cos^2 = 1-\sin^2$. **2.** For tangents, add the two integrals and use $1+\tan^2 = \sec^2$. **3.** Work down to I_0 or I_1 (and T_0 or T_1), which you can do directly. **4.** Check any single value on the calculator.

Section C – Multiple choice (technology active)

Question 6C.1** (1 mark) Linear substitution

[Notes link: **6.4** Linear substitution] Workbook p. 60

Using $u = x+1$, $\int x(x+1)^9\,dx$ becomes

A. $\int (u-1)u^9\,du$ **B.** $\int (u+1)u^9\,du$ **C.** $\int u^{10}\,du$ **D.** $\int xu^9\,du$

Solution. $x = u-1$ and $dx = du$. **A**

Question 6C.2*** (1 mark) Reduction formulae

[Notes link: **6.6** Integration by parts] Workbook p. 60

Let $I_n = \int_0^{\pi/2} \cos^n(x)\,dx$, where $I_n = \dfrac{n-1}{n}I_{n-2}$ for $n \geq 2$. Then I_3 equals

A. $\dfrac{3\pi}{16}$ **B.** $\dfrac{1}{3}$ **C.** $\dfrac{2}{3}$ **D.** $\dfrac{\pi}{4}$

Solution. $I_1 = \int_0^{\pi/2} \cos x\,dx = 1$, so $I_3 = \frac{2}{3}I_1 = \frac{2}{3}$. **C**

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Question 6C.3*** (1 mark) A logarithm in another base

[Notes link: **6.6** Integration by parts; **6.1** Integration formulae and definite integrals] Workbook p. 60

An antiderivative of $\log_2(x)$, for $x > 0$, is

A. $\dfrac{1}{x\log_e(2)}$ **B.** $x\log_2(x) - x$ **C.** $\log_e(2)\,(x\log_e(x) - x)$ **D.** $x\log_2(x) - \dfrac{x}{\log_e(2)}$

Solution. Change the base first: $\log_2(x) = \dfrac{\log_e(x)}{\log_e(2)}$. By parts, $\displaystyle\int \log_e(x)\,dx = x\log_e(x) - x$, so the antiderivative is $\dfrac{x\log_e(x) - x}{\log_e(2)} = x\log_2(x) - \dfrac{x}{\log_e(2)}$. (The first option is the derivative. The second forgets that the $-x$ is divided by $\log_e(2)$ as well, and the third multiplies by $\log_e(2)$ instead of dividing.) **D**

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7 Applications of Integration

Section A – Technology free

Question 7A.1* (5 marks) Integral functions

[Notes link: **7.1** The fundamental theorem and integral functions] Workbook p. 63

Let $F(x) = \int_1^x \sqrt{t^3+1}\,dt$.

a. Find $F(1)$. 1 mark

$F(1) = \int_1^1 \ldots = 0$ **A1**

b. Find $F'(2)$. 1 mark

By the fundamental theorem, $F'(x) = \sqrt{x^3+1}$, so $F'(2) = \sqrt{9} = 3$. **A1**

c. Let $G(x) = \int_0^{x^2} \sin(t)\,dt$. Find $G'(x)$. 2 marks

Let $H(u) = \int_0^u \sin t\,dt$, so $G(x) = H(x^2)$. Chain rule: $G'(x) = \sin(x^2)\cdot 2x = 2x\sin(x^2)$. **M1 A1**

d. Evaluate $\int_0^{\pi} |\cos(x)|\,dx$. 1 mark

$\int_0^{\pi/2} \cos x\,dx - \int_{\pi/2}^{\pi} \cos x\,dx = 1-(-1) = 2$ **A1**

Ways to do it. **1. Fundamental theorem:** $F'(x)$ is the integrand at x, and $F(1) = 0$ because the terminals are equal. **2. Chain rule for a function in the terminal:** $G'(x) = \sin(x^2)\times 2x$. **3. Integrate first, then differentiate,** when the antiderivative is easy ($G(x) = 1-\cos(x^2)$). **4. Modulus:** split at $\frac{\pi}{2}$, or double the first half by symmetry.

Question 7A.2** (7 marks) Volumes about the x-axis

[Notes link: **7.3** Volumes of solids of revolution] Workbook p. 63

Find the exact volume of the solid formed when each region is rotated about the x-axis.

a. The region under $y = \sqrt{x}$ for $0 \le x \le 4$. 2 marks

$\pi\int_0^4 x\,dx = 8\pi$ **M1 A1**

b. The region under $y = \sec(x)$ for $0 \le x \le \frac{\pi}{4}$. 2 marks

$\pi\int_0^{\pi/4} \sec^2 x\,dx = \pi\Big[\tan x\Big]_0^{\pi/4} = \pi$ **M1 A1**

c. The region enclosed by $y = x^2$ and $y = 2x$. 3 marks

Washers with outer radius $2x$ and inner radius x^2: $\pi\int_0^2 \left(4x^2 - x^4\right)dx = \pi\left(\frac{32}{3} - \frac{32}{5}\right) = \frac{64\pi}{15}$ **M1 M1 A1**

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Ways to do it. 1. Discs: $\pi\int y^2\,dx$. **2. Washers:** $\pi\int(R^2-r^2)\,dx$ for a region between two curves, never $\pi\int(R-r)^2$. **3. Standard integral:** $\sec^2 x\to\tan x$. **4. Calculator** to check the value.

Question 7A.3*** (6 marks) Surface areas of revolution

[Notes link: **7.5** Areas of surfaces of revolution] Workbook p. 64

a. The curve $y=\sqrt{x}$, $0\le x\le 2$, is rotated about the x-axis. Find the exact area of the surface formed. 3 marks

$$2\pi\int_0^2\sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx=2\pi\int_0^2\sqrt{x+\frac{1}{4}}\,dx=2\pi\cdot\frac{2}{3}\left[\left(x+\frac{1}{4}\right)^{3/2}\right]_0^2=\frac{4\pi}{3}\left(\frac{27}{8}-\frac{1}{8}\right)=\frac{13\pi}{3}$$ **M1 M1 A1**

b. Show that the surface area of a sphere of radius r is $4\pi r^2$. 3 marks

$y=\sqrt{r^2-x^2}$, $\frac{dy}{dx}=\frac{-x}{\sqrt{r^2-x^2}}$, so $y\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{r^2-x^2}\cdot\frac{r}{\sqrt{r^2-x^2}}=r$. **M1 M1**

$S=2\pi\int_{-r}^{r}r\,dx=4\pi r^2$. **M1**

Ways to do it. 1. Formula: $2\pi\int y\sqrt{1+(y')^2}\,dx$. **2. Simplify before integrating:** $\sqrt{x}\sqrt{1+\frac{1}{4x}}=\frac{1}{2}\sqrt{4x+1}$. **3. Sphere:** for $y=\sqrt{r^2-x^2}$ the product $y\sqrt{1+(y')^2}$ is just r. **4. Parametric check:** $x=r\cos t$, $y=r\sin t$ gives the same integral.

Question 7A.4**** (8 marks) Thinking with the fundamental theorem

[Notes link: **7.1** The fundamental theorem and integral functions] Workbook p. 65

a. Let $F(x)=\int_0^x e^{-t^2}dt$. Explain why F is strictly increasing, and show that $F(-x)=-F(x)$. 2 marks

$F'(x)=e^{-x^2}>0$ for all x, so F is strictly increasing. **M1**

$F(-x)=\int_0^{-x}e^{-t^2}dt$. Let $t=-u$: $=\int_0^x e^{-u^2}(-du)=-F(x)$. **A1**

b. Let $H(x)=\int_x^{2x}\frac{1}{t}\,dt$ for $x>0$. Show that $H'(x)=0$ and explain what this tells you about H. 2 marks

$H'(x)=\frac{1}{2x}\cdot 2-\frac{1}{x}=0$, so H is constant. (In fact $H(x)=\log_e 2x-\log_e x=\log_e 2$.) **M1 A1**

c. Find the area of the region enclosed by $y=\sin^{-1}(x)$, the y-axis and the line $y=\frac{\pi}{2}$. 2 marks

Use $x=\sin y$ for $0\le y\le\frac{\pi}{2}$: area $=\int_0^{\pi/2}\sin y\,dy=1$. **M1 A1**

d. Hence find $\int_0^1\sin^{-1}(x)\,dx$. 2 marks

The rectangle $[0,1]\times\left[0,\frac{\pi}{2}\right]$ splits into the region in part **c.** and the region under $y=\sin^{-1}x$. So $\int_0^1\sin^{-1}x\,dx=\frac{\pi}{2}-1$. **M1 A1**

The trick. The derivative of an integral function is the integrand, so you can find where it increases, and even differentiate it when both terminals move, without ever finding the integral. The area under an inverse function is a rectangle minus

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the area under the original.

Ways to do it. **1. Sign of the integrand:** $F'(x) = e^{-x^2} > 0$, so F increases. **2. Substitution** $u = -t$ to show F is odd. **3. Leibniz rule:** $\frac{d}{dx}\int_{a(x)}^{b(x)} f = f(b)b' - f(a)a'$. **4. Or integrate:** $H(x) = \log_e 2$, a constant. **5. Inverse function area:** rectangle minus the area under the original function, or integrate by parts.

Section B – Technology active

Question 7B.1*** (9 marks) An algorithm for a volume

[Notes link: **7.3** Volumes of solids of revolution; **8.10** Euler's method in pseudocode] Workbook p. 67

The following pseudocode is run.

```
define f(x) = sqrt(x) + 1
left ← 0
total ← 0
h ← 0.5
while left < 2
  r ← f(left + h/2)
  total ← total + π× r²× h
  left ← left + h
end while
print total
```

a. Complete a table showing the values of `left`, `r` and `total` each time the loop runs, and state the value printed, correct to three decimal places. 3 marks

`left` at start	`r`	volume added	`total`
0	$\sqrt{0.25}+1 = 1.5$	3.5343	3.5343
0.5	$\sqrt{0.75}+1 = 1.8660$	5.4697	9.0040
1	$\sqrt{1.25}+1 = 2.1180$	7.0468	16.0508
1.5	$\sqrt{1.75}+1 = 2.3229$	8.4754	24.5262

The loop stops when `left` reaches 2. Printed value: 24.526. **M1 M1 A1**

b. Describe the solid whose volume the algorithm approximates, write down a definite integral for the exact volume, and evaluate it. 3 marks

The solid formed when the region under $y = \sqrt{x}+1$ for $0 \le x \le 2$ is rotated about the x-axis, approximated by four discs of width 0.5 with radius taken at the middle of each strip. **M1 M1**

$$V = \pi\int_0^2 \left(\sqrt{x}+1\right)^2 dx = \pi\int_0^2 \left(x+2\sqrt{x}+1\right) dx = \pi\left(4+\frac{8\sqrt{2}}{3}\right) \approx 24.414$$ **A1**

c. Find the percentage error of the printed value. 1 mark

$$\frac{24.526-24.414}{24.414}\times 100\% \approx 0.46\%$$ **A1**

d. State one change to a single line that would make the approximation more accurate, and say how many times the loop would then run. 2 marks

Make the strips thinner, for example change `h ← 0.5` to `h ← 0.1`. The loop would then run $\frac{2}{0.1} = 20$ times. **M1 A1**

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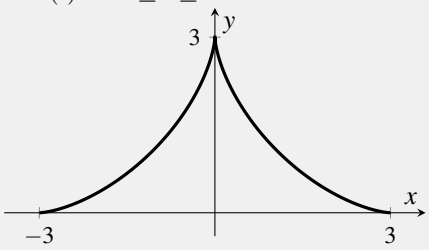
The trick. Trace a loop in a table, one row per pass, and decide exactly when it stops: here the test is `left < 2`, so it does not run for `left = 2`. Then ask what each line is geometrically. $\pi r^2 h$ is a disc, so the total is a volume of revolution.

Ways to do it. **1.** A trace table by hand. **2.** The calculator's sum template: $\sum_{k=0}^{3} \pi\left(\sqrt{0.25+0.5k}+1\right)^2 \times 0.5$. **3.** A Lists & Spreadsheet page with the three columns. **4.** The exact integral for comparison.

Question 7B.2***** (13 marks) A surface where the modulus matters

[Notes link: **7.4** Lengths of curves; **7.5** Areas of surfaces of revolution; **6.7** Integrating a modulus] Workbook p. 68

A curve is defined by $x = 3\cos^3(t),\ y = 3\sin^3(t)$ for $0 \le t \le \pi$.



a. Find $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$. 2 marks

$$\frac{dx}{dt} = -9\cos^2(t)\sin(t), \quad \frac{dy}{dt} = 9\sin^2(t)\cos(t)$$ **M1 A1**

b. Show that $\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2} = 9\left|\sin(t)\cos(t)\right|$. 3 marks

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 81\cos^4 t\sin^2 t + 81\sin^4 t\cos^2 t = 81\sin^2 t\cos^2 t\left(\cos^2 t + \sin^2 t\right) = 81\sin^2 t\cos^2 t.$$ **M1 M1**

The square root is $9\left|\sin(t)\cos(t)\right|$. **M1**

c. Find the length of the curve. 3 marks

On $[0,\pi]$, $\sin t \ge 0$, but $\cos t$ changes sign at $t = \dfrac{\pi}{2}$. **M1 M1**

$$L = \int_0^{\pi} 9\left|\sin t\cos t\right| dt = \int_0^{\frac{\pi}{2}} 9\sin t\cos t\, dt - \int_{\frac{\pi}{2}}^{\pi} 9\sin t\cos t\, dt = \frac{9}{2} + \frac{9}{2} = 9$$ **A1**

d. The curve is rotated about the x-axis. Find the exact area of the surface formed. 4 marks

$$S = 2\pi\int_0^{\pi} |y|\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 2\pi\int_0^{\pi} 3\sin^3 t \times 9\left|\sin t\cos t\right| dt = 54\pi\int_0^{\pi} \sin^4 t\left|\cos t\right| dt.$$ **M1 M1**

Split at $\dfrac{\pi}{2}$; by symmetry the two halves are equal: **M1**

$$S = 108\pi\int_0^{\frac{\pi}{2}} \sin^4 t\cos t\, dt = 108\pi\left[\frac{\sin^5 t}{5}\right]_0^{\frac{\pi}{2}} = \frac{108\pi}{5}$$ **A1**

e. A student leaves out the modulus signs in part **d.** State the answer the student gets, and explain why it cannot be correct. 1 mark

Without the modulus the student evaluates $54\pi\displaystyle\int_0^{\pi} \sin^4 t\cos t\, dt = 54\pi\left[\frac{\sin^5 t}{5}\right]_0^{\pi} = 0$. A surface cannot have zero area: the two halves have cancelled instead of adding. **A1**

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$$2\cdot\pi\cdot\int_0^{\pi}\cdot\left|3\cdot(\sin(t))^3\right|\cdot\sqrt{(\frac{d}{dt}(3(\cos(t))^3))^2+(\frac{d}{dt}(3(\sin(t))^3))^2}\cdot dt \qquad \frac{108\pi}{5}$$

The trick. $\sqrt{a^2}=|a|$. Here $\sqrt{81\sin^2 t\cos^2 t}=9|\sin t\cos t|$, and $\cos t$ is negative for half of the interval. You must split the integral where the inside of the modulus changes sign (Notes 6.7). The warning sign that you forgot is an answer of zero, or a length or area that is smaller than it obviously should be.

Ways to do it. **1.** Parametric formulas with the modulus kept, then split at $t=\frac{\pi}{2}$. **2.** Symmetry: the curve is symmetric about the y-axis, so do $0\le t\le\frac{\pi}{2}$ and double. **3.** Substitution $u=\sin t$ for $\sin^4 t\cos t$. **4.** CAS with abs( ) typed in, which handles the sign change for you.

Section C – Multiple choice (technology active)

Question 7C.1*** (1 mark) A perfect square under the root

[Notes link: **7.4** Lengths of curves] Workbook p. 70

The length of the curve $y=\dfrac{e^x+e^{-x}}{2}$ from $x=0$ to $x=\log_e 2$ is

A. $\frac{1}{2}$ **B.** $\frac{5}{4}$ **C.** $\frac{3}{4}$ **D.** $\log_e 2$

Solution. $1+\left(\frac{e^x-e^{-x}}{2}\right)^2=\left(\frac{e^x+e^{-x}}{2}\right)^2$, so $L=\left[\frac{e^x-e^{-x}}{2}\right]_0^{\log_e 2}=\frac{2-\frac{1}{2}}{2}=\frac{3}{4}$. **C**

Question 7C.2** (1 mark) Surface area about the y-axis

[Notes link: **7.5** Areas of surfaces of revolution] Workbook p. 70

The curve $y=f(x)$, $a\le x\le b$ (with $a\ge 0$), is rotated about the y-axis. The surface area is

A. $2\pi\displaystyle\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$ **B.** $\pi\displaystyle\int_a^b x^2\,dy$

C. $2\pi\displaystyle\int_a^b \sqrt{1+(f'(x))^2}\,dx$ **D.** $2\pi\displaystyle\int_a^b x\sqrt{1+(f'(x))^2}\,dx$

Solution. The radius of each band is the distance from the y-axis, which is x. **D**

Question 7C.3** (1 mark) Integral functions

[Notes link: **7.1** The fundamental theorem and integral functions] Workbook p. 70

If $F(x)=\displaystyle\int_0^x\cos(t^2)\,dt$, then $F'(\sqrt{\pi})$ equals

A. -1 **B.** 1 **C.** 0 **D.** $\cos(\sqrt{\pi})$

Solution. $F'(x)=\cos(x^2)$, so $F'(\sqrt{\pi})=\cos\pi$. **A**

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8 Differential Equations

Section A – Technology free

Question 8A.1* (5 marks) Verifying solutions

[Notes link: **8.1** What is a differential equation?] Workbook p. 73

a. Show that $y = e^{2x} + x$ is a solution of $\dfrac{dy}{dx} - 2y = 1 - 2x$. 2 marks

$\dfrac{dy}{dx} = 2e^{2x} + 1$, so $\dfrac{dy}{dx} - 2y = 2e^{2x} + 1 - 2e^{2x} - 2x = 1 - 2x$. ✓ **M1 M1**

b. Show that $y = \sin(3x)$ is a solution of $\dfrac{d^2y}{dx^2} + 9y = 0$. 1 mark

$\dfrac{d^2y}{dx^2} = -9\sin(3x)$, so $\dfrac{d^2y}{dx^2} + 9y = -9\sin(3x) + 9\sin(3x) = 0$. ✓ **M1**

c. Find the values of k for which $y = e^{kx}$ is a solution of $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6y = 0$. 2 marks

$k^2e^{kx} - ke^{kx} - 6e^{kx} = 0 \Rightarrow k^2 - k - 6 = 0$ (since $e^{kx} \neq 0$) $\Rightarrow k = 3$ or $k = -2$. **M1 A1**

> **Ways to do it. 1. Substitute:** differentiate the given y as many times as needed and put it into both sides; they must agree for every x. **2. Unknown constant:** for $y = e^{kx}$ every term has a factor e^{kx}, leaving a quadratic in k. **3. Calculator:** define y and evaluate left side minus right side; it should simplify to 0.

Question 8A.2** (7 marks) Equations of the form $\frac{dy}{dx} = g(y)$

[Notes link: **8.3** $\dfrac{dy}{dx} = g(y)$: flip it] Workbook p. 74

Solve each differential equation, giving y in terms of x.

a. $\dfrac{dy}{dx} = 2y, \quad y(0) = 3$ 2 marks

$\dfrac{dx}{dy} = \dfrac{1}{2y} \Rightarrow x = \frac{1}{2}\log_e|y| + c$. At $x = 0$, $y = 3$; rearranging gives $y = 3e^{2x}$. **M1 A1**

b. $\dfrac{dy}{dx} = 1 + y^2, \quad y(0) = 1$ 3 marks

$\dfrac{dx}{dy} = \dfrac{1}{1+y^2} \Rightarrow x = \tan^{-1}y + c;\ 0 = \frac{\pi}{4} + c \Rightarrow c = -\frac{\pi}{4}.\ y = \tan\left(x + \dfrac{\pi}{4}\right)$ **M1 M1 A1**

c. $\dfrac{dy}{dx} = e^{-y}, \quad y(0) = 0$ 2 marks

$\dfrac{dx}{dy} = e^y \Rightarrow x = e^y + c;\ 0 = 1 + c \Rightarrow c = -1.\ e^y = x + 1 \Rightarrow y = \log_e(x+1)$ **M1 A1**

> **Ways to do it. 1. Flip it:** $\frac{dx}{dy} = \frac{1}{g(y)}$, integrate with respect to y, then make y the subject. **2. Separate variables:** the same working written as $\displaystyle\int \frac{dy}{g(y)} = \int dx$. **3. Choose the branch** with the condition when a square root or modulus

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appears. **4. Check** by differentiating your answer.

Question 8A.3*** (8 marks) Modelling with differential equations

[Notes link: **8.5** Applications: setting up a differential equation] Workbook p. 75

A cup of coffee at $90°$C is left in a room at $20°$C. Its temperature $T°$C after t minutes satisfies $\dfrac{dT}{dt} = -k(T-20)$, where $k > 0$.

a. Solve the differential equation to find T in terms of t and k. 2 marks

$\displaystyle\int \frac{1}{T-20}\,dT = -\int k\,dt \Rightarrow \log_e(T-20) = -kt + c$. At $t=0$, $T=90$: $T = 20+70e^{-kt}$. **M1 A1**

b. After 10 minutes the coffee is at $55°$C. Find the exact value of k. 2 marks

$55 = 20+70e^{-10k} \Rightarrow e^{-10k} = \frac{1}{2} \Rightarrow k = \dfrac{\log_e 2}{10}$. **M1 A1**

c. Find when the coffee is at $37.5°$C. 1 mark

$37.5 = 20+70e^{-kt} \Rightarrow e^{-kt} = \frac{1}{4} \Rightarrow t = \dfrac{\log_e 4}{k} = 20$ minutes. **A1**

d. A tank holds 100 L of water in which 10 kg of salt is dissolved. Pure water flows in at 5 L/min and the well-mixed solution flows out at 5 L/min. Write down and solve a differential equation for the amount of salt x kg in the tank after t minutes. 3 marks

Concentration $\frac{x}{100}$ kg/L, flowing out at 5 L/min, with no salt coming in: $\dfrac{dx}{dt} = -\dfrac{5x}{100} = -\dfrac{x}{20}$, $x(0) = 10$. **M1 M1**

Separating: $x = 10e^{-t/20}$. **A1**

Ways to do it. 1. Newton's law: separate $\frac{dT}{dt} = -k(T-T_s)$, or quote $T = T_s + Ae^{-kt}$ and find A and k from two data points. **2. Ratio trick:** $\frac{T_2-T_s}{T_1-T_s} = e^{-k\Delta t}$ gives k without finding A first. **3. Mixing:** $\frac{dQ}{dt}$ = (rate in) − (rate out), where rate out $= \frac{Q}{V} \times$ outflow. **4. Long-term value:** set the derivative to zero.

Question 8A.4**** (8 marks) The logistic equation

[Notes link: **8.6** The logistic equation] Workbook p. 76

A population P of fish in a lake after t years satisfies $\dfrac{dP}{dt} = 0.2P\left(1-\dfrac{P}{500}\right)$, with $P(0) = 100$.

a. Show that $\dfrac{1}{P\left(1-\frac{P}{500}\right)} = \dfrac{1}{P} + \dfrac{1}{500-P}$. 1 mark

$\dfrac{1}{P} + \dfrac{1}{500-P} = \dfrac{500-P+P}{P(500-P)} = \dfrac{500}{P(500-P)} = \dfrac{1}{P\left(1-\frac{P}{500}\right)}$ ✓ **M1**

b. Hence show that $P = \dfrac{500}{1+4e^{-0.2t}}$. 4 marks

$\displaystyle\int \left(\frac{1}{P} + \frac{1}{500-P}\right) dP = \int 0.2\,dt \Rightarrow \log_e\left(\frac{P}{500-P}\right) = 0.2t + c$ (for $0 < P < 500$). **M1 M1 M1**

$t=0$: $c = \log_e \frac{100}{400} = -\log_e 4$. So $\dfrac{P}{500-P} = \dfrac{e^{0.2t}}{4} \Rightarrow 4P = 500e^{0.2t} - Pe^{0.2t} \Rightarrow P = \dfrac{500e^{0.2t}}{4+e^{0.2t}} = \dfrac{500}{1+4e^{-0.2t}}$. **M1**

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c. State the limiting population. 1 mark

As $t \to \infty$, $e^{-0.2t} \to 0$, so $P \to 500$. **A1**

d. The population grows fastest when $P = 250$. Find the exact time at which this occurs. 2 marks

$$\frac{500}{1+4e^{-0.2t}} = 250 \Rightarrow 4e^{-0.2t} = 1 \Rightarrow t = 5\log_e 4 = 10\log_e 2 \text{ years.}$$ **M1 A1**

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$\text{deSolve}\left(p' = 0.2 \cdot p \cdot \left(1 - \frac{p}{500}\right) \text{ and } p(0) = 100, t, p\right)$

an equivalent form of $p = \frac{500}{1+4e^{-0.2t}}$

The trick. Read the differential equation before solving it. Setting $\frac{dP}{dt} = 0$ gives the limiting value, and the right-hand side is a parabola in P, so growth is fastest halfway to the limit. The solving itself is partial fractions, with the standard logistic form as a check.

Ways to do it. 1. Separate with partial fractions: $\frac{K}{P(K-P)} = \frac{1}{P} + \frac{1}{K-P}$. **2. Quote and check:** $P = \frac{K}{1+Ae^{-rt}}$ with $A = \frac{K-P_0}{P_0}$. **3. Limiting value:** $\frac{dP}{dt} = 0$ gives $P = K$ without solving anything. **4. Fastest growth:** $\frac{dP}{dt}$ is a parabola in P with its maximum at $P = \frac{K}{2}$. **5. Substitution** $u = \frac{1}{P}$ turns it into a linear equation.

Section B – Technology active

Question 8B.1***** (14 marks) Harvesting

[Notes link: **8.6** The logistic equation; **8.4** Separation of variables; **8.8** Using a definite integral] Workbook p. 77

A fish farm is modelled by $\frac{dP}{dt} = 0.2P\left(1 - \frac{P}{1000}\right) - 32$, where P is the number of fish after t years and 32 fish are removed each year.

a. Show that $\frac{dP}{dt} = -\frac{(P-200)(P-800)}{5000}$. 2 marks

$$0.2P - \frac{P^2}{5000} - 32 = -\frac{P^2 - 1000P + 160\,000}{5000} = -\frac{(P-200)(P-800)}{5000}$$ **M1 M1**

b. State the two equilibrium populations, and describe what happens in the long run when $P(0) = 500$, when $P(0) = 900$ and when $P(0) = 150$. 3 marks

Equilibria: $P = 200$ and $P = 800$. **M1**

$P(0) = 500$: between them $\frac{dP}{dt} > 0$, so P increases towards 800. $P(0) = 900$: $\frac{dP}{dt} < 0$, so P decreases towards 800. **M1**

$P(0) = 150$: $\frac{dP}{dt} < 0$, so P falls, faster and faster, and the fish die out. **A1**

c. For $P(0) = 500$, solve the differential equation to find P in terms of t, and find the time taken for the population to reach 700, correct to two decimal places. 5 marks

$$\int \frac{dP}{(P-200)(P-800)} = -\int \frac{dt}{5000} \Rightarrow \frac{1}{600}\log_e\left|\frac{P-800}{P-200}\right| = -\frac{t}{5000} + c.$$ **M1 M1**

For $200 < P < 800$: $\frac{800-P}{P-200} = Ae^{-0.12t}$, and $P(0) = 500$ gives $A = 1$. **M1**

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$800-P=(P-200)e^{-0.12t} \Rightarrow P=\dfrac{800+200e^{-0.12t}}{1+e^{-0.12t}}$. **M1**

$P=700$: $\dfrac{100}{500}=e^{-0.12t} \Rightarrow t=\dfrac{\log_e(5)}{0.12}=13.41$ years **A1**

d. For $P(0)=150$, write down a definite integral for the time taken for the fish to die out, and evaluate it correct to two decimal places. 2 marks

$$t=\int_{150}^{0}\frac{-5000}{(P-200)(P-800)}\,dP=\int_{0}^{150}\frac{5000}{(P-200)(P-800)}\,dP=9.82 \text{ years}$$ **M1 A1**

e. The number removed each year is changed from 32 to h. Find the largest value of h for which the population can survive. 2 marks

The natural growth $0.2P\left(1-\dfrac{P}{1000}\right)$ has maximum value $0.2\times500\times0.5=50$ at $P=500$. If $h>50$ then $\dfrac{dP}{dt}<0$ for every P and the fish die out. The largest sustainable value is $h=50$. **M1 A1**

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$\displaystyle\int_0^{150}\frac{5000}{(p-200)\cdot(p-800)}\cdot dp$ $\qquad \frac{25}{3}\ln\left(\frac{13}{4}\right)\approx 9.82212$

The trick. The lower equilibrium is a tipping point. Above 200 the population recovers to 800; below 200 it collapses. Work that out from the sign of $\dfrac{dP}{dt}$ before solving anything. For "how long until", flip the equation: $\dfrac{dt}{dP}=\dfrac{1}{dP/dt}$ and integrate between the two populations.

Ways to do it. **1.** Factorise the right-hand side and draw a sign diagram. **2.** Separation with partial fractions for an explicit solution. **3.** A definite integral in P for a time (no need to solve the equation). **4.** Maximum sustainable harvest: the maximum of the growth term, or the discriminant of $0.2P-\frac{P^2}{5000}-h=0$ equal to zero. **5.** CAS deSolve as a check.

Question 8B.2*** (8 marks) A sagging beam

[Notes link: **8.2** $\dfrac{dy}{dx}=f(x)$ and $\dfrac{d^2y}{dx^2}=f(x)$: integrate] Workbook p. 79

A beam of length 4 m rests on supports at its two ends. Its height y metres above the supports, at a horizontal distance x metres from the left end, satisfies $\dfrac{d^2y}{dx^2}=\dfrac{4x-x^2}{100}$ with $y(0)=0$ and $y(4)=0$.

a. Find y in terms of x. 4 marks

$\dfrac{dy}{dx}=\dfrac{1}{100}\left(2x^2-\dfrac{x^3}{3}\right)+c, \quad y=\dfrac{1}{100}\left(\dfrac{2x^3}{3}-\dfrac{x^4}{12}\right)+cx+d.$ **M1 M1**

$y(0)=0\Rightarrow d=0.\ \ y(4)=0\Rightarrow \dfrac{1}{100}\left(\dfrac{128}{3}-\dfrac{64}{3}\right)+4c=0\Rightarrow c=-\dfrac{4}{75}.$ **M1**

$y=\dfrac{8x^3-x^4}{1200}-\dfrac{4x}{75}$ **A1**

b. Find the greatest distance that the beam sags below its supports. 2 marks

$\dfrac{dy}{dx}=0$ at $x=2$ (the middle, by symmetry): $y(2)=\dfrac{48}{1200}-\dfrac{8}{75}=-\dfrac{1}{15}$. The beam sags $\dfrac{1}{15}$ m, about 6.7 cm. **M1 A1**

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c. Find the angle that the beam makes with the horizontal at its left end, in degrees correct to two decimal places. 2 marks

$\frac{dy}{dx} = -\frac{4}{75}$ at $x = 0$, so the angle is $\tan^{-1}\left(\frac{4}{75}\right) = 3.05°$ below the horizontal. **M1 A1**

The trick. A second-order equation needs two constants, and here the two conditions are at different places ($x = 0$ and $x = 4$), not both at the start. Integrate twice, keeping c and d, and only then use the conditions.

Ways to do it. **1.** Integrate twice, then solve for the two constants. **2.** CAS: $\text{deSolve}\left(y'' = \frac{4 \cdot x - x^2}{100} \text{ and } y(0) = 0 \text{ and } y(4) = 0, x, y\right)$. **3.** Use symmetry to locate the lowest point, or solve $\frac{dy}{dx} = 0$.

Section C – Multiple choice (technology active)

Question 8C.1** (1 mark) Setting up a mixing problem

[Notes link: **8.5** Applications: setting up a differential equation] Workbook p. 80

A 100 L tank holds brine containing 5 kg of salt. Brine with 0.2 kg/L flows in at 3 L/min and the mixture flows out at 3 L/min. If x kg is the salt in the tank,

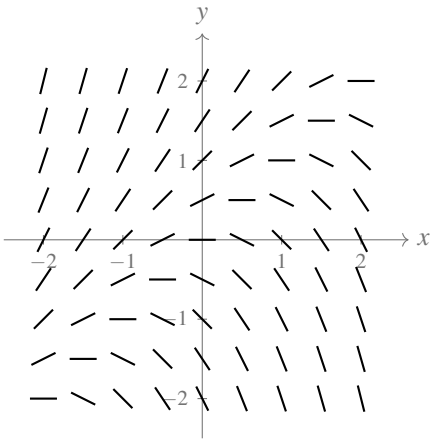
A. $\frac{dx}{dt} = 0.2 - \frac{3x}{100}$ **B.** $\frac{dx}{dt} = 0.6 - \frac{3x}{100}$ **C.** $\frac{dx}{dt} = 0.6 - \frac{x}{100}$ **D.** $\frac{dx}{dt} = 0.6 - \frac{3x}{100+t}$

Solution. In: 0.2×3. Out: concentration $\frac{x}{100}$ times 3 L/min (the volume stays 100). **B**

Question 8C.2*** (1 mark) Matching a slope field by testing points

[Notes link: **8.11** Slope fields] Workbook p. 80

The slope field of a differential equation is shown.



The differential equation could be

A. $\frac{dy}{dx} = x + y$ **B.** $\frac{dy}{dx} = x - y$ **C.** $\frac{dy}{dx} = y - x$ **D.** $\frac{dy}{dx} = xy$

Solution. Test points, one at a time, until one option is left. At $(1, 1)$ the segment is horizontal, so the slope there must be 0: that removes $x + y$ (gives 2) and xy (gives 1). At $(0, 1)$ the segment slopes *up* with gradient 1: $x - y$ gives -1, $y - x$ gives 1. Only $y - x$ fits. Plotting four slope fields on the calculator would take far longer. **C**

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Question 8C.3*** (1 mark) Where the solution lives

[Notes link: **8.4** Separation of variables; **8.3** $\frac{dy}{dx} = g(y)$: flip it] Workbook p. 80

y is the solution of $\frac{dy}{dx} = 3x^2y^2$ with $y(0) = \frac{1}{8}$. The largest set of values of x for which this solution is defined is

A. $x \in R$ **B.** $x < 2$ **C.** $x \in R \setminus \{2\}$ **D.** $-2 < x < 2$

Solution. Separate: $\int y^{-2}\,dy = \int 3x^2\,dx$, so $-\frac{1}{y} = x^3 + c$. With $y(0) = \frac{1}{8}$, $c = -8$ and $y = \frac{1}{8 - x^3}$, which has an asymptote at $x = 2$. A solution of a differential equation is one unbroken curve through the starting point, so it cannot jump across the asymptote: it is the branch containing $x = 0$, which is $x < 2$. ($R \setminus \{2\}$ is the domain of the formula, not of the solution.) **B**

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9 Kinematics

Section A – Technology free

Question 9A.1* (6 marks) Position, velocity and acceleration

[Notes link: **9.1** Position, velocity and acceleration] Workbook p. 83

A particle moves in a straight line so that its position x m from the origin after t seconds is $x = t^3 - 6t^2 + 9t$, $t \geq 0$.

a. Find the velocity of the particle in terms of t. 1 mark

$v = 3t^2 - 12t + 9$ **A1**

b. Find when the particle is at rest. 1 mark

$3(t-1)(t-3) = 0 \Rightarrow t = 1$ and $t = 3$ seconds. **A1**

c. Find the acceleration of the particle when $t = 2$. 1 mark

$a = 6t - 12$, so at $t = 2$, $a = 0\,\text{m/s}^2$. **A1**

d. Find the distance travelled by the particle in the first 4 seconds. 2 marks

The particle turns at $t = 1$ and $t = 3$. $x(0) = 0$, $x(1) = 4$, $x(3) = 0$, $x(4) = 4$. Distance $= 4 + 4 + 4 = 12\,\text{m}$. **M1 A1**

e. Find the average velocity of the particle over the first 4 seconds. 1 mark

$\dfrac{x(4) - x(0)}{4} = 1\,\text{m/s}$ **A1**

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$\int_0^4 \left(\left|3 \cdot t^2 - 12 \cdot t + 9\right|\right) \mathrm{d}t$ $\quad 12$

Ways to do it. **1. Differentiate** x for v, and again for a. **2. Distance:** find when $v = 0$, work out the position at those times and at the ends, and add the pieces. **3. Integral:** $\int |v|\,dt$ on the calculator. **4. Speeding up** when v and a have the same sign.

Question 9A.2** (5 marks) Velocity–time graphs

[Notes link: **9.3** Velocity–time graphs] Workbook p. 84

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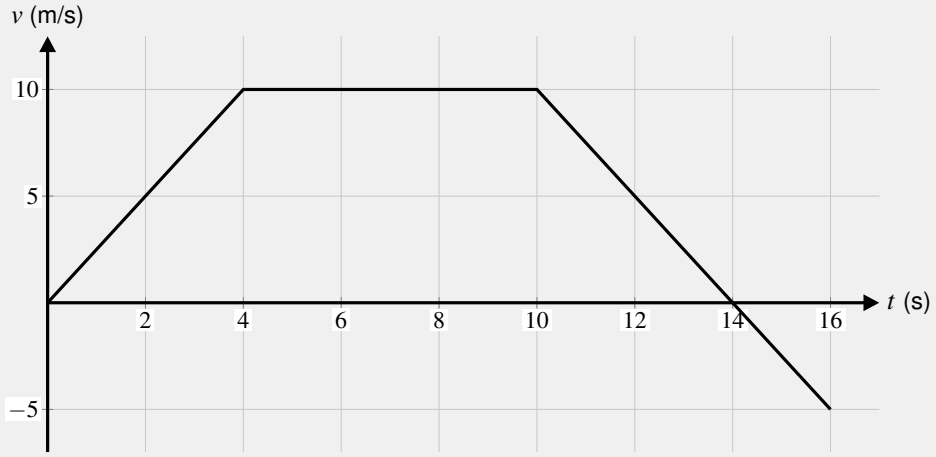
The velocity–time graph of a particle moving in a straight line for $0 \leq t \leq 16$ is shown below.



a. Find the acceleration of the particle during the first 4 seconds. 1 mark

Gradient $= \dfrac{10}{4} = 2.5\,\text{m/s}^2$. **A1**

b. Find the displacement of the particle over the 16 seconds. 2 marks

The velocity is zero at $t = 14$ (the last segment drops 15 over 6 s, i.e. 2.5 per second). **M1**
Signed area $= \frac{1}{2}(4)(10) + 6(10) + \frac{1}{2}(4)(10) - \frac{1}{2}(2)(5) = 20 + 60 + 20 - 5 = 95\,\text{m}$. **A1**

c. Find the distance travelled over the 16 seconds. 1 mark

$20 + 60 + 20 + 5 = 105\,\text{m}$. **A1**

d. At what time is the particle furthest from its starting point? 1 mark

At $t = 14$, when the velocity changes from positive to negative. **A1**

> **Ways to do it. 1. Areas:** signed area is displacement, total area is distance; split into triangles and trapezia. **2. Gradient** is the acceleration. **3. Equations of the line segments,** then integrate, if the shapes are awkward. **4. Average speed** is total distance over total time.

Question 9A.3*** (6 marks) Terminal velocity

[Notes link: **9.1** Position, velocity and acceleration] Workbook p. 85

A particle starts from rest at the origin and moves with acceleration $a = \dfrac{2}{(t+1)^2}\,\text{m/s}^2$.

a. Find the velocity and position of the particle in terms of t. 3 marks

$v = \displaystyle\int \frac{2}{(t+1)^2}\,dt = -\frac{2}{t+1} + c,\ v(0) = 0 \Rightarrow c = 2\text{:}\ v = 2 - \frac{2}{t+1}$. **M1 M1**
$x = 2t - 2\log_e(t+1) + c,\ x(0) = 0 \Rightarrow c = 0\text{:}\ x = 2t - 2\log_e(t+1)$. **A1**

b. State the velocity that the particle approaches as $t \to \infty$. 1 mark

$v \to 2\,\text{m/s}$. **A1**

c. Show that $a = \dfrac{(2-v)^2}{2}$. 2 marks

$2 - v = \dfrac{2}{t+1}$, so $\dfrac{(2-v)^2}{2} = \dfrac{4}{2(t+1)^2} = \dfrac{2}{(t+1)^2} = a$. ✓ **M1 M1**

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Ways to do it. 1. Separate $\frac{dv}{dt} = g - kv$. **2. Terminal velocity:** set $a = 0$; no solving needed. **3. Behaviour:** v approaches the terminal value from below (or above), never reaching it. **4. Distance:** switch to $v\frac{dv}{dx}$.

Question 9A.4**** (8 marks) Resistance proportional to v^2

[Notes link: **9.4** Other expressions for acceleration] Workbook p. 85

A boat cuts its engine when travelling at u m/s. Its acceleration is then $a = -kv^2$, where $k > 0$.

a. Find v in terms of t. 2 marks

$$\int v^{-2}\,dv = -\int k\,dt \Rightarrow -\frac{1}{v} = -kt + c,\ c = -\tfrac{1}{u}.\ \frac{1}{v} = kt + \frac{1}{u} \Rightarrow v = \frac{u}{1+ukt}.$$ **M1 A1**

b. Find v in terms of the distance x travelled since the engine was cut. 2 marks

$$v\frac{dv}{dx} = -kv^2 \Rightarrow \frac{dv}{dx} = -kv \Rightarrow v = ue^{-kx}.$$ **M1 A1**

c. Explain why, according to this model, the boat never stops but travels an unbounded distance. 2 marks

From part **b.**, $v > 0$ for every x, so the boat never stops. From part **a.**, $x = \int_0^t \frac{u}{1+uks}\,ds = \frac{1}{k}\log_e(1+ukt) \to \infty$ as $t \to \infty$. **M1 A1**

d. The boat's speed halves over the first 100 m. Find the exact value of k. 2 marks

$$\tfrac{u}{2} = ue^{-100k} \Rightarrow k = \frac{\log_e 2}{100}.$$ **M1 A1**

The trick. Choose the form of acceleration by what the question asks about: $\frac{dv}{dt}$ when it asks about time, $v\frac{dv}{dx}$ when it asks about distance. With resistance kv^2 the second form loses a v and gives exponential decay in x.

Ways to do it. 1. Time: $\frac{dv}{dt} = -kv^2$, separate to get $\frac{1}{v}$ linear in t. **2. Distance:** $v\frac{dv}{dx} = -kv^2$ gives $\frac{dv}{dx} = -kv$, so $v = ue^{-kx}$. **3. Find** k from the data, then reuse it. **4. Check units** and that the speed never reaches zero in finite distance.

Section B – Technology active

Question 9B.1***** (13 marks) A skydiver in two stages

[Notes link: **9.4** Other expressions for acceleration; **12.14** CAS Guide: Kinematics] Workbook p. 87

A skydiver steps out of a stationary helicopter 1000 m above the ground. For the first 10 seconds her acceleration is $a = 9.8 - 0.2v$, where v m/s is her downward speed. Her parachute then opens instantly, and from then on $a = 9.8 - 1.96v$.

a. Find v in terms of t for $0 \le t \le 10$, and her speed when the parachute opens, correct to two decimal places. 3 marks

$$\frac{dv}{dt} = 0.2(49 - v) \Rightarrow \int \frac{dv}{49-v} = \int 0.2\,dt \Rightarrow -\log_e(49-v) = 0.2t + c;\ v(0) = 0\text{: } c = -\log_e(49).$$ **M1 M1**

$v = 49\left(1 - e^{-0.2t}\right)$, and $v(10) = 49\left(1 - e^{-2}\right) = 42.37$ m/s **A1**

b. Find the distance she falls in the first 10 seconds, correct to two decimal places. 2 marks

$$\int_0^{10} 49\left(1 - e^{-0.2t}\right)dt = 49\left(10 - 5\left(1 - e^{-2}\right)\right) = 278.16\,\text{m}$$ **M1 A1**

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c. State her terminal speed once the parachute is open. 1 mark

$a = 0$ when $v = \dfrac{9.8}{1.96} = 5$ m/s **A1**

d. Find v in terms of t for $t \geq 10$, and how long after the parachute opens her speed first drops to 6 m/s. 3 marks

$\dfrac{dv}{dt} = -1.96(v-5) \Rightarrow v - 5 = Ae^{-1.96(t-10)}$ with $A = 42.37 - 5 = 37.37$. **M1 M1**

$v = 5 + 37.37e^{-1.96(t-10)}$. $v = 6$: $e^{-1.96(t-10)} = \dfrac{1}{37.37} \Rightarrow t - 10 = \dfrac{\log_e(37.37)}{1.96} = 1.85$ s **A1**

e. Find the total time she takes to reach the ground, correct to one decimal place. 3 marks

She has $1000 - 278.16 = 721.84$ m left. With $\tau = t - 10$: **M1**

$\displaystyle\int_0^{\tau} \left(5 + 37.37e^{-1.96s}\right) ds = 5\tau + \frac{37.37}{1.96}\left(1 - e^{-1.96\tau}\right) = 721.84 \Rightarrow \tau = 140.56.$ **M1**

Total time $= 150.6$ s **A1**

f. Find her speed when she lands. 1 mark

$5 + 37.37e^{-1.96 \times 140.56} \approx 5.00$ m/s (she is at terminal speed). **A1**

TI-Nspire | menu 3 1: solve

$v1 := 49 \cdot (1 - \mathbf{e}^{-2})$

$\text{solve}\left(\displaystyle\int_0^x \cdot(5 + (v1 - 5) \cdot \mathbf{e}^{-1.96 \cdot s}) \cdot ds = 1000 - \int_0^{10} \cdot 49 \cdot (1 - \mathbf{e}^{-0.2 \cdot t}) \cdot dt, x\right) | x > 0 \qquad x = 140.555$

The trick. The second stage starts where the first one finished. Its initial speed is the 42.37 m/s from part **a.**, not zero, and its clock starts at $t = 10$. Writing the second stage in terms of $\tau = t - 10$ keeps the algebra clean. Store unrounded values on the calculator.

Ways to do it. **1.** Separate variables for each stage, matching the speed at $t = 10$. **2.** Recognise $\dfrac{dv}{dt} = -k(v - v_T)$: exponential approach to the terminal speed v_T, so $v = v_T + (v_0 - v_T)e^{-kt}$. **3.** Integrate v for distance. **4.** CAS deSolve for each stage and solve for the landing time.

Question 9B.2*** (10 marks) A journey between two stations

[Notes link: **9.3** Velocity–time graphs; **9.2** Constant acceleration] Workbook p. 88

A train starts from rest at one station and accelerates uniformly for 20 seconds until it reaches a speed of V m/s. It travels at this speed for a while, then decelerates uniformly for 10 seconds and stops at the next station. The whole journey takes 100 seconds and covers 1700 m.

a. Sketch the velocity–time graph of the journey on the axes below. 2 marks

A trapezium: straight line from $(0,0)$ up to $(20,V)$, horizontal to $(90,V)$, then straight down to $(100,0)$. **A1 A1**

b. Find V. 3 marks

Distance is the area under the graph. The parallel sides are 100 and 70: $\dfrac{1}{2}(100+70)V = 1700 \Rightarrow V = 20$ m/s **M1 M1 A1**

c. Find the acceleration of the train in the first stage and in the last stage. 2 marks

First stage: $\dfrac{20}{20} = 1$ m/s^2. Last stage: $-\dfrac{20}{10} = -2$ m/s^2. **M1 A1**

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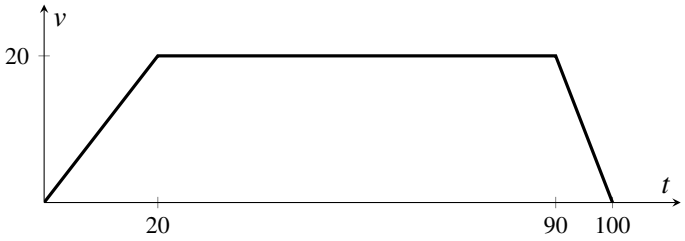
d. Find the time at which the train is halfway between the stations. 2 marks

The first stage covers $\frac{1}{2}(20)(20) = 200$ m. Then $200 + 20(t-20) = 850 \Rightarrow t = 52.5$ s **M1 A1**

e. Find the average speed of the train for the journey. 1 mark

$\frac{1700}{100} = 17$ m/s **A1**



The trick. Draw the graph first and turn the question into geometry: distance is area, acceleration is gradient. One trapezium area equation finds the unknown speed with no kinematics formulas at all. Halfway in distance is not halfway in time.

Ways to do it. **1.** Area under the velocity–time graph (trapezium, or triangle + rectangle + triangle). **2.** Constant acceleration formulas stage by stage: $s = \frac{1}{2}(u+v)t$ for each stage and add. **3.** Integrate a piecewise velocity function on the calculator as a check.

Section C – Multiple choice (technology active)

Question 9C.1*** (1 mark) Resistance proportional to v^2

[Notes link: **9.4** Other expressions for acceleration] Workbook p. 90

A car travelling at 10 m/s hits a barrier and slows with acceleration $-5v^2$ m/s^2. The distance it travels while slowing to 1 m/s is closest to

A. 0.46 m **B.** 0.18 m **C.** 2.30 m **D.** 0.90 m

Solution. $v\frac{dv}{dx} = -5v^2 \Rightarrow \frac{dv}{dx} = -5v \Rightarrow v = 10e^{-5x}$. $1 = 10e^{-5x} \Rightarrow x = \frac{\log_e 10}{5} \approx 0.46$. **A**

Question 9C.2** (1 mark) Time to stop

[Notes link: **9.4** Other expressions for acceleration] Workbook p. 90

A particle moving with velocity 2 m/s has acceleration $a = -(4+v^2)$. The time it takes to come to rest is

A. $\frac{\pi}{4}$ **B.** $\frac{1}{2}$ **C.** $\frac{\pi}{2}$ **D.** $\frac{\pi}{8}$

Solution. $t = \int_2^0 \frac{-1}{4+v^2}\,dv = \left[\frac{1}{2}\tan^{-1}\frac{v}{2}\right]_0^2 = \frac{1}{2}\cdot\frac{\pi}{4}$. **D**

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Question 9C.3*** (1 mark) Speed at the halfway point

[Notes link: **9.2** Constant acceleration] Workbook p. 90

A particle moves in a straight line with constant acceleration. It passes a point A at 2 m/s and later a point B at 14 m/s. Its speed, in m/s, at the point halfway between A and B is

A. 6 **B.** 8 **C.** 10 **D.** 12

Solution. There is no time in the question, so use $v^2 = u^2 + 2as$. Over the whole distance d: $14^2 = 2^2 + 2ad$, so $2ad = 192$. Over half the distance: $v^2 = 2^2 + 2a\left(\frac{d}{2}\right) = 4 + 96 = 100$, so $v = 10$. In general the speed at the midpoint is $\sqrt{\frac{u^2+v^2}{2}}$. (8 is the speed halfway through the *time*; the particle covers the first half of the distance more slowly, so it is going faster than 8 by the time it gets halfway.) **C**

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10 Vector Calculus

Section A – Technology free

Question 10A.1* (7 marks) Vector functions and their paths

[Notes link: **10.1** Vector functions and their paths] Workbook p. 93

The position of a particle at time $t \geq 0$ is $\underset{\sim}{r}(t) = (t-1)\underset{\sim}{i} + t^2\underset{\sim}{j}$.

a. Find the Cartesian equation of the path, stating any restriction on x. 2 marks

$x = t-1 \Rightarrow t = x+1$, so $y = (x+1)^2$. Since $t \geq 0$, $x \geq -1$. **M1 A1**

b. Find the velocity and acceleration of the particle. 2 marks

$\dot{\underset{\sim}{r}}(t) = \underset{\sim}{i} + 2t\underset{\sim}{j}$, $\ddot{\underset{\sim}{r}}(t) = 2\underset{\sim}{j}$. **M1 A1**

c. Find the speed of the particle when $t = 1$. 1 mark

$|\underset{\sim}{i} + 2\underset{\sim}{j}| = \sqrt{5}$ **A1**

d. Sketch the path of the particle on the axes below, showing its starting point and direction of motion. 2 marks

The right half of the parabola $y = (x+1)^2$, starting at $(-1, 0)$ (closed dot) and moving right and up. **A1 A1**

> **Ways to do it. 1. Eliminate t:** substitute, or use $\sin^2 + \cos^2 = 1$. **2. Domain and range of the path** from the range of each component over the allowed t. **3. Direction of motion:** compare two nearby times, or look at the signs of $\dot{x}$ and $\dot{y}$. **4. Differentiate each component** for velocity and acceleration. **5. Calculator:** parametric mode and Graph Trace.

Question 10A.2** (6 marks) Parametric derivatives

[Notes link: **5.8** Differentiating parametric equations] Workbook p. 94

A curve is defined by $x = t^2$, $y = t^3 - 3t$, $t \in R$.

a. Find $\dfrac{dy}{dx}$ in terms of t. 1 mark

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2-3}{2t}$ **A1**

b. Find the coordinates of the points on the curve where the tangent is horizontal. 2 marks

$3t^2 - 3 = 0 \Rightarrow t = \pm 1$: the points $(1, -2)$ and $(1, 2)$. **M1 A1**

c. Find the coordinates of the point on the curve where the tangent is vertical. 1 mark

$\frac{dx}{dt} = 0$ (with $\frac{dy}{dt} \neq 0$) at $t = 0$: the point $(0, 0)$. **A1**

d. Find the equation of the tangent to the curve at the point where $t = 2$. 2 marks

$t = 2$: point $(4, 2)$, gradient $\frac{9}{4}$. $y - 2 = \frac{9}{4}(x-4) \Rightarrow y = \dfrac{9}{4}x - 7$. **M1 A1**

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Ways to do it. 1. $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$**. 2. Horizontal tangent** where $\dot{y} = 0$ and $\dot{x} \neq 0$; **vertical** where $\dot{x} = 0$ and $\dot{y} \neq 0$. **3. Tangent line:** point and gradient at the given t. **4. Convert to Cartesian first** when t eliminates easily.

Question 10A.3*** (5 marks) Distance and displacement

[Notes link: **10.3** Distance travelled and displacement] Workbook p. 94

A particle moves so that $\underset{\sim}{r}(t) = t^2\underset{\sim}{i} + \frac{2}{3}t^3\underset{\sim}{j}, t \geq 0$.

a. Find the distance travelled by the particle from $t = 0$ to $t = \sqrt{3}$. 3 marks

$\dot{\underset{\sim}{r}} = 2t\underset{\sim}{i} + 2t^2\underset{\sim}{j}$, speed $= 2t\sqrt{1+t^2}$. $\int_0^{\sqrt{3}} 2t\sqrt{1+t^2}\,dt = \left[\frac{2}{3}(1+t^2)^{3/2}\right]_0^{\sqrt{3}} = \frac{14}{3}$. **M1 M1 A1**

b. Find the magnitude of the displacement over the same interval, and explain why it is less than your answer to part **a.** 2 marks

$\underset{\sim}{r}(\sqrt{3}) - \underset{\sim}{r}(0) = 3\underset{\sim}{i} + 2\sqrt{3}\underset{\sim}{j}$, with magnitude $\sqrt{21} \approx 4.58$. This is less than $\frac{14}{3} \approx 4.67$ because the path curves: displacement is the straight-line distance between the endpoints. **M1 A1**

Ways to do it. 1. Distance: $\int |\underset{\sim}{v}|\,dt$, simplifying the speed first (look for a perfect square). **2. Displacement:** $\underset{\sim}{r}(b) - \underset{\sim}{r}(a)$, a vector; its magnitude is not the distance. **3. Calculator:** numeric integral of the speed, or arcLen for a Cartesian path.

Question 10A.4**** (8 marks) Collisions and crossing paths

[Notes link: **10.4** Collisions and crossing paths] Workbook p. 95

Two particles move so that $\underset{\sim}{r_A}(t) = (2t-1)\underset{\sim}{i} + (t^2-2)\underset{\sim}{j}$ and $\underset{\sim}{r_B}(t) = (t+1)\underset{\sim}{i} + (3t-4)\underset{\sim}{j}, t \geq 0$.

a. Show that the particles collide, and find when and where. 2 marks

Same t: $2t-1 = t+1 \Rightarrow t = 2$; check y: $2^2 - 2 = 2$ and $3(2) - 4 = 2$. They collide at $t = 2$, at $(3, 2)$. **M1 M1**

b. The paths of the particles also cross at a second point. Find this point, and the time at which each particle passes through it. 3 marks

Paths: A: $y = \left(\frac{x+1}{2}\right)^2 - 2$; B: $y = 3x - 7$. Equating: $(x+1)^2 = 12x - 20 \Rightarrow x^2 - 10x + 21 = 0 \Rightarrow x = 3$ or 7. **M1 M1**

At $(7, 14)$: A passes through it at $t = 4$ and B at $t = 6$, so they cross without colliding. **A1**

c. Find the distance between the particles at $t = 0$. 1 mark

$A(-1, -2)$, $B(1, -4)$: distance $= \sqrt{4+4} = 2\sqrt{2}$. **A1**

d. Find the cosine of the angle between the directions of motion of the particles when they collide. 2 marks

$\dot{\underset{\sim}{r}}_A(2) = 2\underset{\sim}{i} + 4\underset{\sim}{j}$, $\dot{\underset{\sim}{r}}_B = \underset{\sim}{i} + 3\underset{\sim}{j}$: $\cos\theta = \frac{2+12}{\sqrt{20}\sqrt{10}} = \frac{14}{10\sqrt{2}} = \frac{7\sqrt{2}}{10}$. **M1 A1**

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solve$(2\cdot t - 1 = t + 1$ and $t^2 - 2 = 3\cdot t - 4, t)$	$t = 2$
solve$(2\cdot s - 1 = t + 1$ and $s^2 - 2 = 3\cdot t - 4, \{s, t\})$	$s = 2,\ t = 2$ or $s = 4,\ t = 6$

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The trick. Colliding means the same place at the same time. Crossing means the same place at any times. So use one parameter for a collision and two different parameters for the paths.

Ways to do it. 1. Collision: equate both components with the *same* t. **2. Paths cross:** use different parameters t and s and solve. **3. Angle of crossing:** dot product of the two velocity vectors at those times. **4. Closest approach:** minimise $|\underset{\sim}{r}_1 - \underset{\sim}{r}_2|^2$.

Section B – Technology active

Question 10B.1***** (13 marks) Three drones on a cube

[Notes link: **10.2** Velocity, acceleration and speed; **3.8** The cross (vector) product; **3.13** Areas and volumes] Workbook p. 97

Three drones P, Q and R fly along edges of a cube of side 4 units. For $0 \le t \le 4$ their position vectors are

$$\underset{\sim}{r}_P(t) = t\,\underset{\sim}{i}, \qquad \underset{\sim}{r}_Q(t) = 4\,\underset{\sim}{i} + t\,\underset{\sim}{j}, \qquad \underset{\sim}{r}_R(t) = 4\,\underset{\sim}{i} + 4\,\underset{\sim}{j} + t\,\underset{\sim}{k}.$$

a. Find the velocity of each drone, and the angle between the directions of motion of any two of them. 2 marks

$\underset{\sim}{v}_P = \underset{\sim}{i}$, $\underset{\sim}{v}_Q = \underset{\sim}{j}$, $\underset{\sim}{v}_R = \underset{\sim}{k}$. Each pair has dot product 0: the directions are at 90° to each other. **M1 A1**

b. Show that the distance d between P and Q satisfies $d^2 = 2t^2 - 8t + 16$, and find the minimum distance between them. 3 marks

$\overrightarrow{PQ} = (4-t)\,\underset{\sim}{i} + t\,\underset{\sim}{j}$, so $d^2 = (4-t)^2 + t^2 = 2t^2 - 8t + 16 = 2(t-2)^2 + 8$. **M1 M1**

Minimum distance $\sqrt{8} = 2\sqrt{2}$ at $t = 2$. **M1**

c. Show that the area A of triangle PQR satisfies $A^2 = \frac{1}{4}(16-s)(16-3s)$, where $s = t(4-t)$. 4 marks

$\overrightarrow{PR} = (4-t)\,\underset{\sim}{i} + 4\,\underset{\sim}{j} + t\,\underset{\sim}{k}$. With $a = t$ and $b = 4-t$: **M1**

$\overrightarrow{PQ} \times \overrightarrow{PR} = (b\,\underset{\sim}{i} + a\,\underset{\sim}{j}) \times (b\,\underset{\sim}{i} + 4\,\underset{\sim}{j} + a\,\underset{\sim}{k}) = a^2\,\underset{\sim}{i} - ab\,\underset{\sim}{j} + (4b - ab)\,\underset{\sim}{k} = a^2\,\underset{\sim}{i} - ab\,\underset{\sim}{j} + b^2\,\underset{\sim}{k}$ (using $4 - a = b$). **M1**

$A^2 = \frac{1}{4}\left(a^4 + a^2b^2 + b^4\right)$. Now $a^4 + a^2b^2 + b^4 = (a^2+b^2)^2 - a^2b^2 = (a^2 + b^2 + ab)(a^2 + b^2 - ab)$, **M1**

and $a + b = 4$ gives $a^2 + b^2 = 16 - 2ab$. With $s = ab = t(4-t)$: $A^2 = \frac{1}{4}(16-s)(16-3s)$. **M1**

d. Hence find the minimum area of triangle PQR and the time at which it occurs. 3 marks

$(16-s)(16-3s)$ decreases as s increases (for $0 \le s \le 4$ both factors are positive and decreasing). $s = t(4-t)$ is greatest at $t = 2$, where $s = 4$. **M1 M1**

$A^2 = \frac{1}{4}(12)(4) = 12$: minimum area $2\sqrt{3}$ square units at $t = 2$. **A1**

e. State the maximum area of the triangle for $0 \le t \le 4$. 1 mark

s is smallest ($s = 0$) at $t = 0$ and $t = 4$: $A^2 = \frac{1}{4}(16)(16) = 64$, so the maximum area is 8. **A1**

TI-Nspire | menu 4 7: fMin

$p(t) := \begin{bmatrix} t & 0 & 0 \end{bmatrix}$: $q(t) := \begin{bmatrix} 4 & t & 0 \end{bmatrix}$: $r(t) := \begin{bmatrix} 4 & 4 & t \end{bmatrix}$

$\text{fMin}\left(\frac{\text{norm}(\text{crossP}(q(t)-p(t), r(t)-p(t)))}{2}, t\right) \mid 0 \le t \le 4 \qquad t = 2$

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$\dfrac{\text{norm}(\text{crossP}(q(2)-p(2), r(2)-p(2)))}{2}$	$2\sqrt{3}$

The trick. Rename the two lengths that keep appearing: $a = t$ and $b = 4-t$, with $a+b=4$. The cross product and the area then come out in terms of ab only, and the whole problem reduces to "when is $t(4-t)$ biggest?" Minimising A^2 instead of A avoids the square root.

Ways to do it. **1.** Cross product for the area, simplified with $a+b=4$. **2.** Completing the square for the distance. **3.** Monotonic reasoning in s, or differentiate $A^2 = \frac{1}{4}(3s^2-64s+256)$ with respect to t. **4.** CAS fMin on the area function directly. **5.** Check with $t=0$: the triangle is half of a face, area 8.

Question 10B.2*** (10 marks) Looking up at a drone

[Notes link: **10.1** Vector functions and their paths; **10.2** Velocity, acceleration and speed; **3.5** The angle between two vectors; **12.15** CAS Guide: Vector calculus] Workbook p. 98

An observer stands at the origin O on level ground. A drone has position

$$\underset{\sim}{r}(t) = 10t\,\underset{\sim}{i} + (60-5t)\,\underset{\sim}{j} + \left(20+4t-0.2t^2\right)\underset{\sim}{k} \text{ metres}, \quad 0 \le t \le 20,$$

where t is the time in seconds, $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are horizontal and $\underset{\sim}{k}$ is vertically up.

a. Find the position of the drone when $t=4$ and its distance from the observer, correct to one decimal place. 2 marks

$\underset{\sim}{r}(4) = 40\underset{\sim}{i} + 40\underset{\sim}{j} + 32.8\underset{\sim}{k}$. Distance $= |\underset{\sim}{r}(4)| = \sqrt{40^2+40^2+32.8^2} = 65.4$ m. **M1 A1**

b. Find the angle of elevation of the drone from the observer when $t=4$, in degrees correct to one decimal place. 3 marks

The angle of elevation is the angle between $\underset{\sim}{r}$ and the ground. The vector along the ground directly under the drone is $\underset{\sim}{r}$ with its $\underset{\sim}{k}$ component removed: $\underset{\sim}{h} = 40\underset{\sim}{i} + 40\underset{\sim}{j}$. **M1**

$$\cos\theta = \frac{\underset{\sim}{r}\cdot\underset{\sim}{h}}{|\underset{\sim}{r}||\underset{\sim}{h}|} = \frac{1600+1600+0}{65.39\times 56.57} = 0.8651$$ **M1**

$\theta = 30.1°$ **A1**

c. Find the angle of elevation of the drone from the observer at the instant it reaches its greatest height. 2 marks

Height $= 20+4t-0.2t^2$ is greatest when $4-0.4t=0$, $t=10$. $\underset{\sim}{r}(10) = 100\underset{\sim}{i} + 10\underset{\sim}{j} + 40\underset{\sim}{k}$ and $\underset{\sim}{h} = 100\underset{\sim}{i} + 10\underset{\sim}{j}$. **M1**

$\cos\theta = \dfrac{10\,100}{\sqrt{11\,700}\sqrt{10\,100}} = 0.9291$, so $\theta = 21.7°$. **A1**

d. Find the speed of the drone when $t=4$, and the angle its direction of motion makes with the horizontal at that instant. 3 marks

$\underset{\sim}{v}(t) = 10\underset{\sim}{i} - 5\underset{\sim}{j} + (4-0.4t)\underset{\sim}{k}$, so $\underset{\sim}{v}(4) = 10\underset{\sim}{i} - 5\underset{\sim}{j} + 2.4\underset{\sim}{k}$. Speed $= \sqrt{100+25+5.76} = 11.4$ m/s. **M1**

Horizontal part of the velocity: $10\underset{\sim}{i} - 5\underset{\sim}{j}$. $\cos\alpha = \dfrac{125}{\sqrt{130.76}\sqrt{125}} = 0.9777$ **M1**

$\alpha = 12.1°$ above the horizontal (the $\underset{\sim}{k}$ component is positive, so it is climbing). **A1**

TI-Nspire | menu 7 C: Vector, and angle in degree mode

$r := [40 \quad 40 \quad 32.8]$	$[40 \quad 40 \quad 32.8]$
$h := [40 \quad 40 \quad 0]$	$[40 \quad 40 \quad 0]$
$\cos^{-1}\left(\dfrac{\text{dotP}(r,h)}{\text{norm}(r)\cdot\text{norm}(h)}\right)$	30.1063

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The trick. An angle of elevation (or a climb angle) is the angle between a vector and the horizontal. The easiest way to get it is a dot product of the vector with *itself minus its $\underset{\sim}{k}$ component*. No diagram and no right-angled triangle to set up: copy the vector, set the last component to 0, and use the angle formula. Because $\underset{\sim}{r}\cdot\underset{\sim}{h}=|\underset{\sim}{h}|^2$, it even simplifies to $\cos\theta=\dfrac{|\underset{\sim}{h}|}{|\underset{\sim}{r}|}$.

Watch out. The greatest angle of elevation does not happen at the greatest height: at $t=10$ the drone is higher than at $t=4$ but much further away, and the angle is smaller. Elevation depends on height *and* horizontal distance.

Ways to do it. **1. Dot product with the horizontal copy** of the vector. **2. Right-angled triangle:** $\tan\theta=\dfrac{z}{\sqrt{x^2+y^2}}=\dfrac{32.8}{56.57}$. **3. Angle with the vertical:** $\cos\phi=\dfrac{\underset{\sim}{r}\cdot\underset{\sim}{k}}{|\underset{\sim}{r}|}$, then $\theta=90^\circ-\phi$. **4. Line and plane formula:** the ground is a plane with normal $\underset{\sim}{k}$, so $\sin\theta=\dfrac{|\underset{\sim}{r}\cdot\underset{\sim}{k}|}{|\underset{\sim}{r}|}$. **5. Calculator:** store the vector, copy it with the last entry 0, then dotP and norm; or pull out components with $r[1,3]$.

Section C – Multiple choice (technology active)

Question 10C.1** (1 mark) Projectile

[Notes link: **10.2** Velocity, acceleration and speed] Workbook p. 100

A projectile has position $\underset{\sim}{r}(t)=20t\,\underset{\sim}{i}+(20t-4.9t^2)\,\underset{\sim}{j}$ metres. Its maximum height is closest to

A. 20.4 m **B.** 40.8 m **C.** 81.6 m **D.** 4.1 m

Solution. $\dot{y}=0$ at $t=\frac{20}{9.8}$; $y=\frac{20^2}{2(9.8)}\approx 20.4$. **A**

Question 10C.2*** (1 mark) Projectile: the angle of impact

[Notes link: **10.1** Vector functions and their paths; **10.2** Velocity, acceleration and speed] Workbook p. 100

A ball is hit from a point 1 m above level ground. Its position t seconds later is $\underset{\sim}{r}(t)=14t\,\underset{\sim}{i}+\left(1+9.8t-4.9t^2\right)\,\underset{\sim}{j}$ metres, where $\underset{\sim}{i}$ is horizontal and $\underset{\sim}{j}$ is vertically up. The acute angle that the ball's path makes with the ground at the instant it lands is closest to

A. 35.0° **B.** 37.5° **C.** 52.5° **D.** 55.0°

Solution. It lands when $1+9.8t-4.9t^2=0$, $t=2.097$. Then $\underset{\sim}{v}=14\,\underset{\sim}{i}+(9.8-9.8t)\,\underset{\sim}{j}=14\,\underset{\sim}{i}-10.75\,\underset{\sim}{j}$, and $\tan^{-1}\left(\frac{10.75}{14}\right)=37.5^\circ$. ($35.0^\circ$ is the launch angle: it would be the answer only if the ball landed at the height it was hit from. 52.5° is the angle with the vertical.) **B**

Question 10C.3*** (1 mark) Acceleration at right angles to velocity

[Notes link: **10.2** Velocity, acceleration and speed; **3.4** The dot (scalar) product] Workbook p. 100

A particle has position $\underset{\sim}{r}(t)=k\sin(t)\,\underset{\sim}{i}+t^2\,\underset{\sim}{j}$, where k is a positive constant. Its acceleration is perpendicular to its velocity when $t=\dfrac{\pi}{4}$. The value of k is

A. $\sqrt{\dfrac{\pi}{2}}$ **B.** $\sqrt{\pi}$ **C.** 2 **D.** $\sqrt{2\pi}$

Solution. $\underset{\sim}{v}=k\cos(t)\,\underset{\sim}{i}+2t\,\underset{\sim}{j}$ and $\underset{\sim}{a}=-k\sin(t)\,\underset{\sim}{i}+2\,\underset{\sim}{j}$. $\underset{\sim}{v}\cdot\underset{\sim}{a}=-k^2\sin(t)\cos(t)+4t=-\dfrac{k^2}{2}\sin(2t)+4t$. At $t=\dfrac{\pi}{4}$: $-\dfrac{k^2}{2}+\pi=0$, so $k^2=2\pi$ and $k=\sqrt{2\pi}$. (When the acceleration is perpendicular to the velocity, the speed is momentarily not changing.) **D**

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11 Probability and Statistics

Section A – Technology free

Question 11A.1* (6 marks) Linear combinations of random variables

[Notes link: **11.1** Linear combinations of random variables] Workbook p. 103

X and Y are independent random variables with $\mathrm{E}(X)=5$, $\mathrm{Var}(X)=4$, $\mathrm{E}(Y)=3$ and $\mathrm{Var}(Y)=9$.

a. Find $\mathrm{E}(2X-3)$ and $\mathrm{Var}(2X-3)$. 2 marks

$\mathrm{E}(2X-3)=2(5)-3=7;\ \mathrm{Var}(2X-3)=2^2(4)=16.$ **M1 A1**

b. Find $\mathrm{E}(X-Y)$ and $\mathrm{Var}(X-Y)$. 2 marks

$\mathrm{E}(X-Y)=5-3=2;\ \mathrm{Var}(X-Y)=4+9=13$ (variances add, even for a difference). **M1 A1**

c. X_1, X_2 and X_3 are independent observations of X. Find $\mathrm{Var}(X_1+X_2+X_3)$ and explain why it is not equal to $\mathrm{Var}(3X)$. 2 marks

$\mathrm{Var}(X_1+X_2+X_3)=3(4)=12$, but $\mathrm{Var}(3X)=9(4)=36$. $3X$ is one observation tripled, so its variation is tripled too; independent observations partly cancel each other out. **M1 A1**

Ways to do it. 1. Rules: $\mathrm{E}(aX+b)=a\,\mathrm{E}(X)+b$ and $\mathrm{Var}(aX+b)=a^2\,\mathrm{Var}(X)$; the $+b$ does nothing to the variance. **2. Independent variables:** $\mathrm{Var}(X-Y)=\mathrm{Var}(X)+\mathrm{Var}(Y)$; variances add even for a difference. **3. One thing scaled or several things added?** $\mathrm{Var}(3X)=9\,\mathrm{Var}(X)$ but $\mathrm{Var}(X_1+X_2+X_3)=3\,\mathrm{Var}(X)$. **4. Standard deviations never add:** work in variances and take the root at the end.

Question 11A.2** (6 marks) Confidence intervals

[Notes link: **11.4** Confidence intervals for the mean] Workbook p. 104

A random sample of 100 batteries has a mean lifetime of 52 hours. The population standard deviation is known to be 10 hours. Use $z=1.96$ for 95% confidence.

a. Find a 95% confidence interval for the mean lifetime μ. 2 marks

$52\pm1.96\times\dfrac{10}{\sqrt{100}}=52\pm1.96$, i.e. $(50.04,\ 53.96)$ hours. **M1 A1**

b. Explain what is meant by "95% confidence" in this context. 1 mark

If many samples of 100 were taken and an interval built from each, about 95% of those intervals would contain the true mean μ. (It does *not* mean there is a 95% chance that μ is in this particular interval.) **A1**

c. Find the smallest sample size needed for the margin of error of a 95% confidence interval to be at most 1 hour. 2 marks

$1.96\times\dfrac{10}{\sqrt{n}}\le1\Rightarrow\sqrt{n}\ge19.6\Rightarrow n\ge384.16$, so $n=385$. **M1 A1**

d. State how a 99% confidence interval from the same sample would compare with the interval in part **a.** 1 mark

Wider, because the z-value is larger (2.576 instead of 1.96). **A1**

Ways to do it. 1. Formula: $\bar{x}\pm z\frac{\sigma}{\sqrt{n}}$. **2. Sample size:** $n\ge\left(\frac{z\sigma}{M}\right)^2$, always rounded up. **3. Scaling:** the width is proportional to $\frac{1}{\sqrt{n}}$, so four times the sample halves it. **4. Calculator:** zInterval.

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Question 11A.3**** (8 marks) Combining discrete random variables

[Notes link: **11.1** Linear combinations of random variables] Workbook p. 104

A random variable X has the probability distribution below.

x	0	1	2
$\Pr(X=x)$	0.2	0.5	0.3

a. Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. 3 marks

$\mathrm{E}(X) = 0(0.2)+1(0.5)+2(0.3) = 1.1$. $\mathrm{E}(X^2) = 0.5+1.2 = 1.7$. $\mathrm{Var}(X) = 1.7-1.21 = 0.49$. **M1 M1 A1**

b. Y is the sum of 100 independent observations of X. State the approximate distribution of Y, with its mean and standard deviation. 2 marks

By the central limit theorem, Y is approximately normal with mean $100\times 1.1 = 110$ and sd $\sqrt{100\times 0.49} = 7$. **M1 A1**

c. Use the fact that about 95% of a normal distribution lies within two standard deviations of the mean to estimate $\Pr(Y > 124)$. 1 mark

$124 = 110+2\times 7$, so $\Pr(Y>124) \approx \frac{1-0.95}{2} = 0.025$. **A1**

d. Find $\mathrm{E}(X_1-2X_2)$ and $\mathrm{Var}(X_1-2X_2)$, where X_1 and X_2 are independent observations of X. 2 marks

$\mathrm{E}(X_1-2X_2) = 1.1-2.2 = -1.1$; $\mathrm{Var}(X_1-2X_2) = 0.49+4(0.49) = 2.45$. **M1 A1**

The trick. Build everything from one observation. Find the mean and variance of a single one from the table, then scale: a total of n observations has mean $n\mu$ and variance $n\sigma^2$, and any multiplier is squared in the variance.

Ways to do it. 1. Table: $\mathrm{E}(X) = \sum xp$ and $\mathrm{Var}(X) = \mathrm{E}(X^2)-\mathrm{E}(X)^2$. **2. Sum of n observations:** mean $n\mu$, variance $n\sigma^2$. **3. Coefficients are squared** in a variance. **4. Large n:** the total is approximately normal.

Question 11A.4*** (4 marks) The normal approximation to the binomial

[Notes link: **11.3** The distribution of the sample mean] Workbook p. 106

A fair coin is tossed 100 times. Let X be the number of heads. You may use $\Pr(Z<2) = 0.977$.

a. State the approximate distribution of X, with its mean and standard deviation. 2 marks

$X \sim \mathrm{Bi}(100, 0.5)$, approximately $N(50, 5^2)$: mean $np = 50$, sd $\sqrt{np(1-p)} = 5$. **M1 A1**

b. Estimate $\Pr(X \geq 60)$, and explain why the approximation is reasonable. 2 marks

$\Pr(X\geq 60) \approx \Pr(Z \geq 2) = 0.023$. Reasonable because n is large and $p = 0.5$, so the distribution is symmetric and np and $n(1-p)$ are both large. **M1 A1**

Ways to do it. 1. Parameters: mean np, variance $np(1-p)$. **2. Normal approximation** when np and $n(1-p)$ are both large. **3. Compare** with the exact $\mathrm{binomCdf}(\,)$ value.

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Section B – Technology active

Question 11B.1*** (6 marks) Sample means from a density function

[Notes link: **11.2** Revision: continuous random variables; **11.3** The distribution of the sample mean] Workbook p. 107

X has probability density function $f(x) = \dfrac{x}{2}$ for $0 \le x \le 2$ (and 0 otherwise).

a. Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. 3 marks

$\mathrm{E}(X) = \displaystyle\int_0^2 \frac{x^2}{2}\,dx = \frac{4}{3};\ \mathrm{E}(X^2) = \int_0^2 \frac{x^3}{2}\,dx = 2;\ \mathrm{Var}(X) = 2 - \tfrac{16}{9} = \frac{2}{9}.$ **M1 M1 A1**

b. State the approximate distribution of the mean $\bar{X}$ of a random sample of 50 values of X. 1 mark

By the central limit theorem, $\bar{X} \sim N\left(\frac{4}{3},\ \frac{2/9}{50}\right) = N\left(\frac{4}{3}, \left(\frac{1}{15}\right)^2\right)$ approximately. **A1**

c. Find $\Pr(\bar{X} > 1.4)$, correct to three decimal places. 2 marks

$\Pr(\bar{X} > 1.4) = \Pr\left(Z > \frac{1.4-4/3}{1/15}\right) = \Pr(Z > 1) \approx 0.159.$ **M1 A1**

TI-Nspire | Calculator

$\displaystyle\int_0^2 \frac{x\cdot x}{2}\,\mathrm{d}x\ :\ \int_0^2 \frac{x^2\cdot x}{2}\,\mathrm{d}x$ $\quad \frac{4}{3},\quad 2$

$\mathrm{normCdf}\left(1.4, \infty, \dfrac{4}{3}, \dfrac{1}{15}\right)$ $\quad 0.158655$

Ways to do it. 1. Integrate for $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$, then the variance. **2. Sample mean:** approximately $N\left(\mu, \frac{\sigma^2}{n}\right)$ for a large sample, whatever the shape of the density. **3. Calculator:** $\mathrm{normCdf}(\ )$.

Question 11B.2***** (15 marks) Finding the true mean

[Notes link: **11.6** Errors in hypothesis testing; **11.5** Hypothesis testing for the mean] Workbook p. 108

A council claims that the mean time it takes to repair a pothole is 12 days. Repair times are normally distributed with standard deviation 4 days. A residents' group believes the mean is longer, and tests H_0: $\mu = 12$ against H_1: $\mu > 12$ at the 1% level of significance using a random sample of n repairs.

a. The group will reject H_0 if the sample mean is more than 13.163 days (correct to three decimal places). Find n. 2 marks

Under H_0, $\Pr(\bar{X} > 13.163) = 0.01 \Rightarrow \dfrac{13.163 - 12}{4/\sqrt{n}} = 2.3263 \Rightarrow \sqrt{n} = \dfrac{4 \times 2.3263}{1.163} = 8.00$, so $n = 64$. **M1 A1**

b. With this test, the probability of a Type II error is 0.2503. Find the true mean repair time, correct to one decimal place. 3 marks

The standard deviation of $\bar{X}$ is $\dfrac{4}{\sqrt{64}} = 0.5$. A Type II error is not rejecting, $\bar{X} < 13.163$, when the mean is really μ_T: **M1 M1**

$\Pr(\bar{X} < 13.163 \mid \mu = \mu_T) = 0.2503 \Rightarrow \dfrac{13.163 - \mu_T}{0.5} = -0.6735 \Rightarrow \mu_T = 13.163 + 0.337 = 13.5$ days. **A1**

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c. The group doubles its sample size and keeps the same level of significance. Find the new critical value, correct to three decimal places, and the new probability of a Type II error, correct to four decimal places. 3 marks

With $n = 128$ the standard deviation of $\bar{X}$ is $\dfrac{4}{\sqrt{128}} = 0.35355$. **M1 M1**

$c_2 = 12 + 2.3263 \times 0.35355 = 12.822$. $\Pr(\bar{X} < 12.822 \mid \mu = 13.5) = 0.0277$ **A1**

d. A student is told only the level of significance, the standard deviation, and that the probability of a Type II error is 0.2503 for a sample of size n and 0.0277 for a sample of size $2n$. The student is *not* told n, the critical values c_1 and c_2, or the true mean μ_T, and tries to find all four.

Write down the four equations the student can form. Give values from the standard normal distribution correct to four decimal places. 4 marks

Size n, if H_0 is true: $\dfrac{c_1 - 12}{4/\sqrt{n}} = 2.3263$ (1) Size n, true mean: $\dfrac{c_1 - \mu_T}{4/\sqrt{n}} = -0.6735$ (2) **M1 M1 M1**

Size $2n$, if H_0 is true: $\dfrac{c_2 - 12}{4/\sqrt{2n}} = 2.3263$ (3) Size $2n$, true mean: $\dfrac{c_2 - \mu_T}{4/\sqrt{2n}} = -1.9157$ (4) **A1**

e. By eliminating c_1 from one pair of equations and c_2 from the other pair, show that the four equations cannot be solved for μ_T. Give two different pairs of values of n and μ_T that fit all of the information. 3 marks

(1) − (2): $\dfrac{(\mu_T - 12)\sqrt{n}}{4} = 2.3263 + 0.6735 = 3.000$, so $(\mu_T - 12)\sqrt{n} = 12.00$. **M1**

(3) − (4): $\dfrac{(\mu_T - 12)\sqrt{2n}}{4} = 2.3263 + 1.9157 = 4.242$, so $(\mu_T - 12)\sqrt{n} = \dfrac{4 \times 4.242}{\sqrt{2}} = 12.00$. **M1**

Both pairs give the *same* equation. So there is really only one equation for the two unknowns n and μ_T, and it has many solutions: any pair with $(\mu_T - 12)\sqrt{n} = 12$ fits.

For example $n = 64$, $\mu_T = 13.5$ and $n = 144$, $\mu_T = 13$ (also $n = 36$, $\mu_T = 14$). **A1**

TI-Nspire | menu 5 5 3: invNorm and menu 5 5 2: normCdf

$\text{invNorm}(0.99, 0, 1)$	2.32635
$\text{invNorm}(0.2503, 0, 1)$	-0.673546
$\text{invNorm}(0.0277, 0, 1)$	-1.91573
$c2 := \text{invNorm}\left(0.99, 12, \dfrac{4}{\sqrt{128}}\right)$	12.8225
$\text{normCdf}\left(-\infty, c2, 13.5, \dfrac{4}{\sqrt{128}}\right)$	0.027664
$\text{normCdf}\left(-\infty, \text{invNorm}\left(0.99, 12, \dfrac{4}{12}\right), 13, \dfrac{4}{12}\right)$	0.250266

The trick. A critical value ties the test to a sample size, and a Type II probability then ties it to the true mean: one equation each, in that order. Part **e.** is the catch. The probability of a Type II error depends on n and μ_T only through the single number $\dfrac{(\mu_T - \mu_0)\sqrt{n}}{\sigma}$, so knowing it for n and again for $2n$ tells you the same thing twice. That is why a calculator refuses to solve the four equations: they are not four independent equations. You need one more fact, such as the critical value in part **a.**, before the true mean can be found.

Ways to do it. **1.** Standardise each statement and solve one unknown at a time. **2.** invNorm for the z-value of a given area, normCdf to check. **3.** Subtract paired equations to eliminate a critical value. **4.** Test your understanding of part **e.** by checking that $n = 144$, $\mu_T = 13$ gives 0.2503 and 0.0277 as well (last line of the calculator box). **5.** A sketch of the two curves for $\bar{X}$.

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Section C – Multiple choice (technology active)

Question 11C.1** (1 mark) A confidence interval

[Notes link: **11.4** Confidence intervals for the mean] Workbook p. 110

A sample of 36 croissants has mean length 15 cm. The population variance is 2 cm^2. An approximate 95% confidence interval for the mean length is

A. $(14.35,\ 15.65)$ **B.** $(14.54,\ 15.46)$ **C.** $(14.67,\ 15.33)$ **D.** $(12.23,\ 17.77)$

Solution. $15 \pm 1.96 \times \frac{\sqrt{2}}{6} = 15 \pm 0.46$. (A uses $\sigma = 2$ by mistake.) **B**

Question 11C.2*** (1 mark) Sample mean of a continuous variable

[Notes link: **11.3** The distribution of the sample mean; **11.2** Revision: continuous random variables] Workbook p. 110

X has probability density function $f(x) = 2x$ for $0 \le x \le 1$ (and 0 otherwise). For random samples of size 50, the standard deviation of $\bar{X}$ is

A. $\frac{1}{50}$ **B.** $\frac{\sqrt{2}}{6}$ **C.** $\frac{1}{30}$ **D.** $\frac{1}{900}$

Solution. $\text{E}(X) = \frac{2}{3}$, $\text{E}(X^2) = \frac{1}{2}$, so $\text{Var}(X) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$. Then $\text{sd}(\bar{X}) = \sqrt{\frac{1}{18 \times 50}} = \frac{1}{30}$. **C**

Question 11C.3** (1 mark) A p-value

[Notes link: **11.5** Hypothesis testing for the mean] Workbook p. 110

Boxes are claimed to have mean mass 400 g, with $\sigma = 10$ g. A sample of 25 boxes has mean 396 g. For $H_1 : \mu < 400$, the p-value is closest to

A. 0.0456 **B.** 0.4207 **C.** 0.9772 **D.** 0.0228

Solution. $z = \frac{396-400}{10/5} = -2$, so $p = \Pr(Z < -2)$. (A is the two-tailed value.) **D**

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